Passage-time moments and hybrid zones for the exclusion-voter model

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Abstract

We study the non-equilibrium dynamics of a one-dimensional interacting particle system that is a mixture of the voter model and exclusion process. With the process started from a finite perturbation of the ground-state Heaviside configuration consisting of 1s to the left of the origin and 0s elsewhere, we study the relaxation time τ, that is, the first hitting time of the ground-state configuration (up to translation). We give conditions for τ to be finite and for certain moments of τ to be finite or infinite, and prove a result that approaches a conjecture of Belitsky *et al.* [*Bernoulli* **7** (2001) 119–144]. Ours are the first non-existence of moments results for τ for the mixture model. Moreover, we give almost-sure asymptotics for the evolution of the size of the hybrid (disordered) region. Most of our results pertain to the discrete-time setting, but several transfer to continuous-time. As well as the mixture process, some of our results also cover pure exclusion. We state several significant open problems.

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1 Introduction

The *exclusion-voter* model is a one-dimensional lattice-based interacting-particle process with nearest-neighbour interactions, introduced by Belitsky *et al.* in [7], that is a mixture of the symmetric voter model and the simple exclusion process. For background on the latter two models (separately), and interacting particle systems in general, see [16, 17].

The voter model has been used to model the spread of an opinion through a static population via nearest-neighbour interactions; see e.g. [13]. The mixture model studied here is a natural extension of this model, whereby individuals do not have to remain static, but may move by switching places. Alternative motivation, such as from the point of view of competition of species (see e.g. [8]) also can be adapted to the mixture model. As our results show, allowing place-swaps can have a dramatic effect on the dynamics of the process.

The exclusion-voter model is a Markov process with state-space $\{0, 1\}^{\mathbb{Z}}$; each site of $\mathbb{Z}$ can be labelled either 0 or 1, representing the presence of one of two types of particle. The ground-state of our model will be the 'Heaviside' configuration $\ldots 111000 \ldots$. We consider initial configurations that are finite perturbations of this ground-state, and so contain a finite number of unlike pairs, where by 'pair' we always mean two adjacent particles.

In this paper, we concentrate on a *discrete time* process that can be described informally as follows. At each time step, the *simple exclusion process* selects uniformly at random from amongst all unlike pairs. If the chosen pair is 01, it flips to 10 with probability p (else there is no change); if the pair is 10, it flips to 01 with probability $1 - p$. On the other hand, at each time step the *symmetric voter model* selects uniformly at random from all unlike pairs and then flips the chosen pair to either 00 or 11, with equal chance of each. The model that is considered in this paper, introduced in [7], is a mixture of these two processes, whereby at each time step we determine independently at random whether to perform a voter-type move (with probability β) or an exclusion-type move (probability $1 - \beta$).

The analogous continuous-time exclusion-voter model can be defined via its infinitesimal generator and constructed via a Harris-type graphical construction. The discrete-time process described above is naturally embedded in the continuous-time process. In our analysis we work in discrete-time, and the discrete-time process has its own interest, but, as we shall indicate, some of our results transfer almost immediately into continuous time.

Individually, the exclusion process and voter model exhibit very different behaviour. For instance, in the exclusion process there is local conservation of 1s: the number of 1s in a bounded interval can change only through the boundary. There is no such conservation in the voter model. In the mixture process that we study in the present paper, voter moves and exclusion moves interact in a highly non-trivial way. This introduces technical difficulties: for instance, voter moves can cause drastic changes quickly, and there is no obvious monotonicity property. We describe the model more formally and state our results in the next section. First we outline the existing literature and the contribution of the present paper.

In [7], results were proved for the exclusion process and voter model separately, as well as some initial results for the mixture model. The main problems left open in [7] were the non-existence of passage-time moments and the issue of transience/recurrence for the mixture model. As we describe shortly, the present paper makes contributions to each of these problems. Some of the results in [7], in the symmetric exclusion ($p = 1/2$) case, are generalized to non-nearest-neighbour interactions in [20]. Certain 'ergodic' properties of a generalization of the continuous-time exclusion-voter model, again in the symmetric exclusion case, are studied in [14]. The goal of the present paper is to study the mixture model in more depth than [7]. In particular we prove new results on: (i) the passage-time problem for the exclusion-voter model, the main contribution being the (more difficult) non-existence of passage-time moments; and (ii) the size of the disordered region where 1s and 0s intermingle. This region we call the *hybrid zone* (cf. [9]). Our results leave several open problems, and we make some conjectures with regard to these in the next section.

Let us describe more specifically the contribution of the present paper to the passage-time problem for the exclusion-voter model. The passage-time of interest to us here is the *relaxation time* τ — the return time of the configuration to the ground-state. In general, often one can prove the existence of moments of passage-times directly via semimartingale (Lyapunov-type function) criteria such as those in [2, 4, 15] in the vein of Foster [12]. The non-existence of moments (for which no results were previously obtained for the exclusion-voter model with $\beta \in (0, 1)$) is usually a harder problem. In general, semimartingale-type arguments are available in this case too (see e.g. [3, 4, 15]), but under more restrictive conditions than the corresponding existence results: non-existence results typically need fine control over jumps of the process. Lamperti [15] was first to establish a general methodology for proving non-existence of passage-time moments, based upon finding a suitable submartingale and obtaining a good-probability lower bound for passage times; his method was later extended in [3, 4]. The same two elements form the basis of our approach, but we must proceed differently since the exclusion-voter model does not possess the regularity required by existing general results such as those of [3, 4, 15].

On the one hand, we extend the region of the parameter space of the model for which almost-sure finiteness of τ is known, and we give results on the existence of higher moments of τ (including in the case of pure exclusion). On the other hand, we show the *non-existence* of certain moments of τ; this problem was not addressed in [7]. Each of these opposed directions requires us to develop new techniques. We prove, for example, that under certain conditions $1 + \varepsilon$ moments ($\varepsilon > 0$) of τ do not exist; this approaches a conjecture in [7].

The second main contribution of the paper is to study the evolution of the size of the hybrid zone. Our basic tools are again semimartingales: we apply general results on almost-sure bounds for stochastic processes from [18]. For instance, for the pure exclusion process in the case $p = 1/2$ we prove that with probability 1 the maximum size of the hybrid zone up to time t remains bounded between $t^{1/3}$ and $t^{1/2}$, ignoring logarithmic factors.

In the next section we give some more formal definitions, state our main results, and discuss some (challenging) open problems.

2 Definitions and statement of results

We now formally describe the model that we study, as considered in [7]. We introduce some notation to describe the configuration of the process. Let $\mathcal{D}' \subset \{0, 1\}^{\mathbb{Z}}$ denote the set of configurations with a finite number of 0s to the left of the origin and 1s to the right. Let '$\sim$' denote the equivalence relation on $\mathcal{D}'$ such that for $S, S' \in \mathcal{D}'$, $S \sim S'$

if and only if S and S' are translates of each other. Then set $\mathcal{D} := \mathcal{D}'/\sim$. In other words, the configuration space $\mathcal{D}$ is the set of configurations of the form infinite string of 1s—finite number of 0s and 1s—infinite string of 0s, modulo translations. For example, one configuration $S \in \mathcal{D}$ is

$$S = \ldots 111000000000111000010010010000000001111000\ldots. \tag{1}$$

Configurations such as those in $\mathcal{D}$ are sometimes called *shock profiles* (see e.g. [7]).

Fix $\beta \in [0,1]$ (the mixing parameter) and $p \in [0,1]$ (the exclusion parameter). The discrete-time exclusion-voter process $\xi = (\xi_t)_{t\in\mathbb{Z}^+}$ with parameters (β, p) is a time-homogeneous Markov chain on the countable state-space $\mathcal{D}$. The one-step transition probabilities are determined by the following mechanism. At each time step we decide independently at random whether to perform a *voter* move or an *exclusion* move. We choose a voter move with probability β and an exclusion move with probability $1-\beta$. Having decided this, choose an unlike adjacent pair (i.e. 01 or 10) uniformly at random. The voter move is such that the chosen pair (01 or 10) flips to 00 or 11 each with probability 1/2. The exclusion move is such that a chosen pair 01 flips to 10 with probability p (otherwise no move) and a chosen pair 10 flips to 01 with probability $q := 1-p$ (otherwise no move).

In addition to the discrete-time model that is the focus of the present paper, there is a corresponding continuous-time model, also introduced in [7]. A priori, the relationship between the two time-scales is complicated, but from our results on the discrete-time process we can obtain some results in the continuous-time setting too. For a description of the continuous-time model, its relationship to the discrete-time model that is our main object of study, and our results in the continuous-time setting, see Section 3 below.

The underlying probability space for ξ we denote by $(\Omega, \mathcal{F}, \mathbb{P}_{\beta,p})$, and the corresponding expectation $\mathbb{E}_{\beta,p}$. We denote the ground-state Heaviside configuration $\mathcal{D}_0 \in \mathcal{D}$, which consists of a single pair 10 abutted by infinite strings of 1s and 0s to the left and right, respectively:

$$\mathcal{D}_0 = \ldots 11110000\ldots,$$

up to translation. The next result gives some elementary properties of the state-space $\mathcal{D}$ under $\mathbb{P}_{\beta,p}$. In particular, Proposition 1 says that for $(\beta,p) \in (0,1)^2$ (i.e., in the interior of the parameter space) ξ is irreducible and aperiodic under $\mathbb{P}_{\beta,p}$.

Proposition 1 *$\mathcal{D}_0$ is an absorbing state under $\mathbb{P}_{\beta,1}$ for any $\beta \in [0,1]$. Suppose $\beta \neq 1$ and $(\beta,p) \notin \{(0,0),(0,1)\}$. Then all states in $\mathcal{D} \setminus \{\mathcal{D}_0\}$ communicate under $\mathbb{P}_{\beta,p}$. Suppose $\beta \neq 1$, $p < 1$, and $(\beta,p) \neq (0,0)$. Then all states in $\mathcal{D}$ communicate under $\mathbb{P}_{\beta,p}$, and ξ is irreducible and aperiodic.*

For $S_0 \in \mathcal{D}$ define the *relaxation time* for the process ξ as

$$\tau := \min\{t \in \mathbb{N} : \xi_t = \mathcal{D}_0\}.$$

We introduce some convenient terminology. If $\mathbb{P}_{\beta,p}(\tau = +\infty \mid \xi_0 = S_0) > 0$ for $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, we say that ξ is *transient* started from S_0; if $\mathbb{P}_{\beta,p}(\tau < \infty \mid \xi_0 = S_0) = 1$ for $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, we say that ξ is *recurrent* started from S_0. In the latter case, if in addition $\mathbb{E}_{\beta,p}[\tau \mid \xi_0 = S_0] < \infty$ for $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, we say that ξ is *positive-recurrent* started from S_0. When ξ is irreducible (see Proposition 1), this terminology coincides with the standard usage for countable state-space Markov chains. When ξ is irreducible and aperiodic (see Proposition 1), we may use the term *ergodic* in the positive-recurrent case.

Results of Liggett (see e.g. Chapter VIII of [16]) imply that the pure exclusion process ($\beta = 0$) is positive-recurrent for all $S_0 \in \mathcal{D}$ if and only if $p > 1/2$. We recall the following result, which is contained in Theorems 5.1, 5.2, 6.1, 7.1, and 7.2 of [7], together with an inspection of (7.2) in [7] for part (iii)(a).

Theorem 1 *(i) Suppose $\beta = 0$ (pure exclusion). Then for any $S_0 \in \mathcal{D}$, ξ is positive-recurrent for $p > 1/2$ and transient for $p \leq 1/2$.*

(ii) Suppose $\beta = 1$ (pure voter). Then ξ is positive-recurrent for any $S_0 \in \mathcal{D}$, and moreover for any $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ and any $\varepsilon > 0$,

$$\mathbb{E}_{1,p}[\tau^{(3/2)-\varepsilon} \mid \xi_0 = S_0] < \infty; \quad \mathbb{E}_{1,p}[\tau^{(3/2)+\varepsilon} \mid \xi_0 = S_0] = \infty.$$

(iii) Suppose $\beta \in (0, 1)$ (mixture process).

(a) If β and $p \in [0, 1]$ are such that $(1 - p)(1 - \beta) < 1/3$, then ξ is positive-recurrent for any $S_0 \in \mathcal{D}$. In particular, for any $\beta > 2/3$ and any $p \in [0, 1]$, ξ is positive-recurrent for any $S_0 \in \mathcal{D}$.

(b) For $p \geq 1/2$ and any $\beta > 0$, ξ is positive-recurrent for any $S_0 \in \mathcal{D}$.

In [7], the following was Conjecture 7.1.

Conjecture 1 *For any $p < 1/2$, there exists $\beta_0 = \beta_0(p) > 0$ such that for any $\beta < \beta_0$, ξ is not positive-recurrent, i.e. $\mathbb{E}_{\beta,p}[\tau \mid \xi_0 = S_0] = \infty$ for any $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$.*

Our first result says that for β small enough (so that the exclusion part is prevalent), $1 + \varepsilon$ moments do not exist; thus Conjecture 1 remains tantalizingly open.

Theorem 2 *For each $p < 1/2$, there exists $\beta_1 = \beta_1(p) = (1 - 2p)/(2 - 2p) \in (0, 1/2]$ such that for all $\beta \leq \beta_1$, for any $\varepsilon > 0$, for any $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$,*

$$\mathbb{E}_{\beta,p}[\tau^{1+\varepsilon} \mid \xi_0 = S_0] = \infty.$$

Our second result says that in the mixture process, the presence of a transient exclusion ensures that $2 + \varepsilon$ moments do not exist. Thus for $p \leq 1/2$, even in the case where Theorem 1(iii) applies the recurrence is polynomial in nature, i.e. 'heavy-tailed'.

Theorem 3 *Suppose $p \leq 1/2$, $\beta \in [0, 1]$. For any $\varepsilon > 0$ and $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$,*

$$\mathbb{E}_{\beta,p}[\tau^{2+\varepsilon} \mid \xi_0 = S_0] = \infty.$$

In view of Theorem 1(ii), we suspect that mixing transient ($p \leq 1/2$) exclusion with the voter model ought not lead to a lighter tail for τ:

Conjecture 2 *Suppose $p \leq 1/2$, $\beta \in [0, 1]$. For any $\varepsilon > 0$ and $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$,*

$$\mathbb{E}_{\beta,p}[\tau^{(3/2)+\varepsilon} \mid \xi_0 = S_0] = \infty.$$

Even this conjecture seems to be challenging, as exclusion and voter moves interact in complex ways. Technically, the issue that prevents us from reducing the 2 to 3/2 in Theorem 3 is that exclusion moves can (and typically will) increase the number of blocks.

An open problem mentioned in [7] is whether the mixture process with $\beta > 0$ and $p < 1/2$ is in fact transient (it is recurrent for $p \geq 1/2$, by Theorem 1(iii)(b)). Simulations that we have performed have been inconclusive. We conjecture the following.

Conjecture 3 *Suppose $p < 1/2$, $\beta > 0$. For any $S_0 \in \mathcal{D}$, ξ is recurrent.*

Note that if Conjectures 1 and 3 both hold, there is *null-recurrence* for $p < 1/2$ and $\beta \in (0, \beta_0)$. Our next result represents some progress in the direction of Conjecture 3, and gives recurrence in a previously unexplored region of the parameter space.

Theorem 4 *Suppose $p < 1/2$, $\beta \geq 4/7$. For any $S_0 \in \mathcal{D}$, ξ is recurrent.*

Now we turn to the problem of existence of moments for τ. First we consider the pure exclusion process in the positive-recurrent ($p > 1/2$) case. If we further restrict to $p > 2/3$, it is possible to construct a positive strict supermartingale with uniformly bounded increments (see (5.7) in [7]), and so it is not hard to show that all polynomial moments of τ exist in that case. Theorem 5 below extends this conclusion to all $p > 1/2$.

Theorem 5 *Suppose $\beta = 0$, $p > 1/2$. For any $S_0 \in \mathcal{D}$ and any $s \in [0, \infty)$*

$$\mathbb{E}_{0,p}[\tau^s \mid \xi_0 = S_0] < \infty.$$

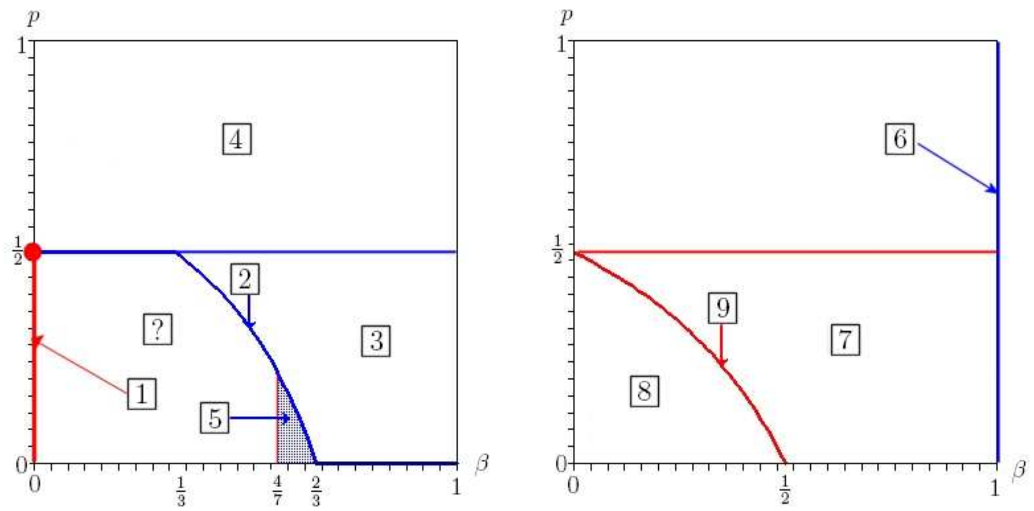
We suspect that, under the conditions of Theorem 5, the existence of some super-polynomial 'moments' for τ can be obtained via our techniques and general results from [3]. The next result covers the mixture process in the case where the exclusion component is positive-recurrent. In the $\beta \in [0, 1]$, $p > 1/2$ case we know from Theorem 1 that $\mathbb{E}[\tau] < \infty$; the next theorem says that some higher moments are finite also.

Theorem 6 *Suppose $\beta \in [0, 1]$, $p > 1/2$. For any $S_0 \in \mathcal{D}$, $\mathbb{E}_{\beta,p}[\tau^{6/5} \mid \xi_0 = S_0] < \infty$.*

In view of Theorem 1 and Theorem 5, in the setting of Theorem 6 we are mixing together the voter model, for which $(3/2) - \varepsilon$ moments exist, and the recurrent exclusion process, for which all moments exist. Thus one might hope to improve the exponent in Theorem 6 to at least $(3/2) - \varepsilon$: this is another open problem.

Figure 1 gives two diagrams of the (β, p) parameter space, summarizing the results of the previous theorems for the relaxation time τ.

We now state our results on the size of the hybrid zone. First we need to introduce some more notation, following [7]. A 1-block (0-block) is a maximal string of consecutive 1s (0s). Configurations in $\mathcal{D}$ consist of a finite number of such blocks. For $S \in \mathcal{D}$, let $N = N(S) \geq 0$ denote the number of 1-blocks not including the infinite 1-block to the left (this is the same as number of 0-blocks not including the infinite 0-block to the right). Enumerating left to right, let $n_i = n_i(S)$ denote the size of the i-th 0-block, and $m_i = m_i(S)$ the size of the i-th 1-block. We may represent configuration $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ by the vector $(n_1, m_1, \ldots, n_N, m_N)$. For example, the configuration S of (1), which has $N(S) = 5$, has the representation $(8, 3, 4, 1, 2, 1, 2, 1, 8, 4)$. Set $|\mathcal{D}_0| := 0$ and for $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ let $|S| := \sum_{i=1}^{N}(n_i + m_i)$ the size of the hybrid zone, i.e., the length



1	$\mathbb{P}(\tau = \infty) > 0$ (Theorem 1(i))	6	$\mathbb{E}[\tau^{(3/2)+\varepsilon}] = \infty$ and $\mathbb{E}[\tau^{(3/2)-\varepsilon}] < \infty$ for $\varepsilon > 0$ (Theorem 1(ii))
2	The curve $(1-p)(1-\beta) = 1/3$ for $\beta \in [1/3, 2/3]$	7	$\mathbb{E}[\tau^{2+\varepsilon}] = \infty$ (Theorem 3)
3	$\mathbb{E}[\tau] < \infty$ (Theorem 1(iii))	8	$\mathbb{E}[\tau^{1+\varepsilon}] = \infty$ (Theorem 2)
4	$\mathbb{E}[\tau^{6/5}] < \infty$ (Theorem 6)	9	The curve $\beta = (1-2p)/(2-2p)$ for $p \in [0, 1/2]$
5	$\mathbb{P}(\tau < \infty) = 1$ (Theorem 4)	?	Recurrent? (Conjecture 3)

Figure 1: Representations of the (β, p) parameter space. The key given explains the labelling together with the appropriate result from the text (for brevity we have dropped the subscripts on $\mathbb{P}, \mathbb{E}$ in the table).

of the string of 0s and 1s between the infinite string of 1s to the left and the infinite string of 0s to the right.

The next result gives upper bounds for the size of the hybrid zone $|\xi_t|$ and the number of blocks $N(\xi_t)$; in particular, part (ii) covers the case $\beta = 0$, $p = 1/2$ of the symmetric pure (transient) exclusion process.

Theorem 7 *(i) Suppose $\beta \in [0,1]$, $p \in [0,1]$. For any $\varepsilon > 0$, $\mathbb{P}_{\beta,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$,*

$$\max_{0 \le s \le t} N(\xi_s) \le \begin{cases} t^{1/2}(\log t)^{(1/2)+\varepsilon} & \text{if } p < 1/2 \\ t^{1/3}(\log t)^{(1/3)+\varepsilon} & \text{if } p \ge 1/2 \end{cases}. \tag{2}$$

(ii) Suppose $\beta \in [0,1]$, $p \ge 1/2$. For any $\varepsilon > 0$, $\mathbb{P}_{\beta,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$,

$$\max_{0 \le s \le t} |\xi_s| \le t^{1/2}(\log t)^{(1/2)+\varepsilon}. \tag{3}$$

The remainder of our results deal with the pure exclusion process ($\beta = 0$). In the continuous-time setting, related results on the growth of the hybrid zone of the pure exclusion process were first obtained by Rost [19] in the totally asymmetric case; see Section VIII.5 of [16], and [1] for more general results. In particular, Theorems 5.2, 5.3, and 5.12 on pages 403–407 of [16] say, very loosely, that under $\mathbb{P}_{0,p}$

$$|\eta_t| \approx t \;\; (p < 1/2); \qquad |\eta_t| \approx t^{1/2} \;\; (p = 1/2),$$

where η is the continuous-time version of ξ as described in Section 3. In particular, the symmetric case is significantly different from the asymmetric case. However, there seems to be no immediate way to translate these results between the continuous- and discrete-time settings (see Section 3 below). Part (i) of the next result strengthens the bound in (2) slightly in the pure exclusion case with $p < 1/2$. Part (ii) complements the $\beta = 0$ case of (3) for the case $p < 1/2$ (transient but not symmetric exclusion); it quantifies the rate of transience.

Theorem 8 *Suppose $\beta = 0$ and $p \in [0, 1]$.*

(i) There exists $C \in (0, \infty)$ such that for any $p \in [0, 1]$, $\mathbb{P}_{0,p}$-a.s., for all $t \in \mathbb{Z}^+$,

$$\max_{0 \leq s \leq t} N(\xi_s) \leq C t^{1/2}.$$

(ii) Suppose $p \in [0, 1/2)$. Then for any $\varepsilon > 0$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$

$$\max_{0 \leq s \leq t} |\xi_s| \leq t^{2/3} (\log t)^{(1/3)+\varepsilon}.$$

On the other hand, there exists $c(p) \in (0, \infty)$ such that for any $c \in (0, c(p))$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$

$$|\xi_t| \geq c t^{1/2}.$$

Our next result complements (3) in the case $\beta = 0$, $p = 1/2$.

Theorem 9 *Suppose $\beta = 0$, $p = 1/2$. For any $\varepsilon > 0$, $\mathbb{P}_{0,1/2}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$,*

$$t^{1/3} (\log t)^{-(1/3)-\varepsilon} \leq \max_{0 \leq s \leq t} |\xi_s| \leq t^{1/2} (\log t)^{(1/2)+\varepsilon}.$$

It is an open problem to obtain sharper versions of the above results on $|\xi_t|$. In the pure exclusion ($\beta = 0$) case, we conjecture the following.

Conjecture 4 *Suppose $\beta = 0$. If $p < 1/2$, then for any $\varepsilon > 0$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$, $|\xi_t| \leq t^{(1/2)+\varepsilon}$. If $p = 1/2$, then for any $\varepsilon > 0$, $\mathbb{P}_{0,1/2}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$, $t^{(1/3)-\varepsilon} \leq |\xi_t| \leq t^{(1/3)+\varepsilon}$.*

The structure of the remainder of the paper is as follows. In Section 3 we describe the *continuous time* version of the exclusion-voter model, and how it relates to the discrete-time version studied here, and which results can be transferred without too much extra work. Section 4 contains preliminary results: in Section 4.1 we collect general semimartingale results that we apply in the paper, in Section 4.2 we introduce notation and a convenient representation for configurations of the model and prove Proposition 1, and in Section 4.3 we give some lemmas on the Lyapunov-type functions that we will use throughout the paper. In Section 5 we prove Theorems 2 and 3 on passage-time moments, via a series of lemmas. In Section 6 we prove Theorem 4. In Section 7 we prove Theorems 5 and 6. In Section 8 we prove Theorems 7, 8, and 9 on the size of the hybrid zone and number of blocks.

3 Continuous time

The exclusion-voter model may also be defined and studied in continuous time; first we recall the definition, following [7]. Let $\nu = (\nu(x))_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, so that $\nu(x)$ is the label (0 or 1) at x. For $x, y, z \in \mathbb{Z}$ denote

$$\nu_{x,y}(z) := \begin{cases} \nu(y) & \text{if } z = x, \\ \nu(x) & \text{if } z = y, \\ \nu(z) & \text{if } z \neq x, y; \end{cases} \qquad \nu_x(z) := \begin{cases} 1 - \nu(z) & \text{if } z = x, \\ \nu(z) & \text{if } z \neq x. \end{cases}$$

In words, $\nu_{x,y}$ is ν with labels at x, y interchanged, and ν_x is ν with the label at x flipped (i.e., replaced by its opposite). We introduce Markovian generators Ω^{e}_p ($p \in [0,1]$) and Ω^{v} defined by their action on functions f on $\{0,1\}^{\mathbb{Z}}$ by:

$$\begin{aligned} \Omega^{\mathrm{e}}_p f(\nu) &= \sum_{x,y} p(x,y)\nu(x)(1-\nu(y))[f(\nu_{x,y}) - f(\nu)], \text{ and} \\ \Omega^{\mathrm{v}} f(\nu) &= \sum_{x} c(x,\nu)[f(\nu_x) - f(\nu)], \end{aligned}$$

where $p(x, x-1) = p$, $p(x, x+1) = 1 - p$ and $p(x, y) = 0$ for $|x - y| \neq 1$, and

$$c(x,\nu) := \begin{cases} \frac{1}{2}(\nu(x-1) + \nu(x+1)) & \text{if } \nu(x) = 0, \\ \frac{1}{2}(2 - \nu(x-1) - \nu(x+1)) & \text{if } \nu(x) = 1. \end{cases}$$

The continuous-time exclusion-voter model with mixing parameter $\beta \in [0,1]$ and exclusion parameter $p \in [0,1]$ is a Markov process $(\eta'_t)_{t\geq 0}$ on $\mathcal{D}' \subset \{0,1\}^{\mathbb{Z}}$ with generator $(1-\beta)\Omega^{\mathrm{e}}_p + \beta\Omega^{\mathrm{v}}$. This induces a Markov process $\eta = (\eta_t)_{t\geq 0}$ on the space of equivalence classes $\mathcal{D}$ by taking η_t to be the $\sim$-equivalence class of η'_t. The process η can be constructed from an array of homogeneous one-dimensional Poisson processes via a Harris-type graphical construction: see p. 9 of [7] for details. With the definitions in Section 2 and this section, ξ may be embedded in η in the standard way: again, see [7].

In the continuous-time setting, the relaxation time is

$$\tau_{\mathrm{c}} := \inf\{t \geq 0 : \eta_t = \mathcal{D}_0\}.$$

The natural question is: given the results in Section 2 on τ, what is it possible to say about τ_{c}? We now outline which of our discrete-time results for ξ can be readily transferred to continuous-time results for η. Compare Section 8 of [7].

First of all, as pointed out in [7], recurrence and transience transfer directly:

$$\mathbb{P}_{\beta,p}(\tau < \infty \mid \xi_0 = S_0) = 1 \iff \mathbb{P}_{\beta,p}(\tau_{\mathrm{c}} < \infty \mid \eta_0 = S_0).$$

To conclude about moments (i.e., tails) of the relaxation times it is necessary to know about the comparative rates of the two processes. The transition rate of the continuous-time process is, roughly speaking, proportional to the number of blocks, so the continuous-time process tends to evolve at least as fast as the discrete-time process.

The pure voter model ($\beta = 1$) is well-behaved in the sense that it cannot increase the number of blocks. Thus, roughly speaking, the discrete and continuous timescales are directly comparable, and results are more easily transferred. This intuition is formalized in Section 8 of [7], where it is shown that for any $s > 0$

$$\mathbb{E}_{1,p}[\tau^s \mid \xi_0 = S_0] < \infty \iff \mathbb{E}_{1,p}[\tau_{\mathrm{c}}^s \mid \eta_0 = S_0] < \infty.$$

In the general case, without more information on the number of blocks, only one-sided results are possible a priori. It is shown in Section 8 of [7] that for any $s > 0$

$$\mathbb{E}_{\beta,p}[\tau^s \mid \xi_0 = S_0] < \infty \Longrightarrow \mathbb{E}_{\beta,p}[\tau_{\rm c}^s \mid \eta_0 = S_0] < \infty.$$

Thus Theorem 1 above (proved in [7]) transfers directly to continuous time, and holds with $\tau_{\rm c}$ instead of τ; this is Theorem 1.1 in [7]. In particular, the $\beta = 1$ case of this result shows that the continuous-time pure voter model is positive-recurrent, a result that goes back to Cox and Durrett (Theorem 4 of [9]); for further study of voter model interfaces and some generalizations see [5, 6, 10]. Moreover our Theorems 4, 5 and 6 also carry across, and hold with $\tau_{\rm c}$, yielding the following corollary.

Corollary 1 *For any $S_0 \in \mathcal{D}$, we have the following:*

(i) Suppose $p < 1/2$, $\beta \geq 4/7$. Then η is recurrent, i.e., $\mathbb{P}_{\beta,p}(\tau_{\rm c} < \infty \mid \eta_0 = S_0) = 1$.

(ii) Suppose $\beta = 0$, $p > 1/2$. Then, for any $s \in [0, \infty)$, $\mathbb{E}_{0,p}[\tau_{\rm c}^s \mid \xi_0 = S_0] < \infty$.

(iii) Suppose $\beta \in [0, 1]$, $p > 1/2$. Then $\mathbb{E}_{\beta,p}[\tau_{\rm c}^{6/5} \mid \xi_0 = S_0] < \infty$.

Corollary 1(ii) says that for the standard (continuous-time) recurrent exclusion process, all moments of $\tau_{\rm c}$ exist. This fact may be known, but we could not find a reference.

4 Preliminaries

4.1 Technical tools

In this section we state some general martingale-type results that we will need. In particular, we will recall some criteria for obtaining upper and lower almost sure bounds for discrete-time stochastic processes on the half-line given in [18].

Let $(\mathcal{F}_t)_{t\in\mathbb{Z}^+}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X = (X_t)_{t\in\mathbb{Z}^+}$ be a discrete time $(\mathcal{F}_t)$-adapted stochastic process taking values in $[0, \infty)$. Suppose that $\mathbb{P}(X_0 = x_0) = 1$ for some $x_0 \in [0, \infty)$. For the applications in the present paper, we will for instance take $X_t = |\xi_t|$. The following result combines a maximal inequality (Lemma 3.1 in [18]) with an almost-sure upper bound (contained in Theorem 3.2 of [18]).

Lemma 1 *Let $B \in (0, \infty)$ be such that, for all $t \in \mathbb{Z}^+$,*

$$\mathbb{E}[X_{t+1} - X_t \mid \mathcal{F}_t] \leq B \text{ a.s.}. \tag{4}$$

Then: (i) for any $r > 0$ and any $t \in \mathbb{N}$

$$\mathbb{P}\left(\max_{0\leq s\leq t} X_s \geq r\right) \leq (Bt + x_0)r^{-1}; \tag{5}$$

and (ii) for any $\varepsilon > 0$, a.s., for all but finitely many $t \in \mathbb{Z}^+$,

$$\max_{0\leq s\leq t} X_s \leq t(\log t)^{1+\varepsilon}.$$

We also state the following result on existence of passage-time moments for one-dimensional stochastic processes, which is a simple consequence of Theorem 1 of [4].

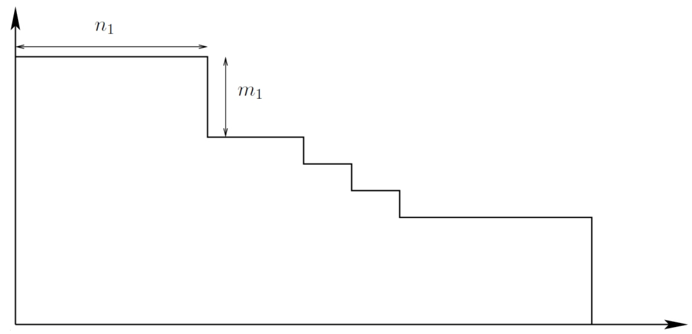


Figure 2: An example staircase configuration.

Lemma 2 *Let* $(X_t)_{t\in\mathbb{Z}^+}$ *be an* $(\mathcal{F}_t)_{t\in\mathbb{Z}^+}$*-adapted stochastic process taking values in an unbounded subset* $\mathcal{S}$ *of* $[0,\infty)$*. Suppose* $B>0$*. Set* $\upsilon_B := \min\{t\in\mathbb{N} : X_t \le B\}$*. Suppose that there exist* $C\in(0,\infty)$*,* $\gamma\in[0,1)$ *such that for any* $t\in\mathbb{Z}^+$

$$\mathbb{E}[X_{t+1}-X_t \mid \mathcal{F}_t] \le -CX_t^{\gamma} \quad \text{on } \{\upsilon_B > t\}.$$

Then for any $p\in[0,1/(1-\gamma)]$*, for any* $x\in\mathcal{S}$*,* $\mathbb{E}[\upsilon_B^p \mid X_0 = x] < \infty$*.*

4.2 Exclusion-voter configurations

We introduce some more notation. For $S\in\mathcal{D}\setminus\{\mathcal{D}_0\}$ and $i\in\{1,\dots,N\}$ let

$$R_i := R_i(S) := \sum_{j=1}^{i} n_j, \quad \text{and} \quad T_i := T_i(S) := \sum_{j=i}^{N} m_j. \tag{6}$$

It is convenient to represent a configuration $S\in\mathcal{D}\setminus\{\mathcal{D}_0\}$ diagrammatically as a right-down path in the quarter-lattice $\mathbb{Z}^+\times\mathbb{Z}^+$: starting from $(0,T_1)$, construct a walk by reading left-to-right the configuration S and for each 0 (1) taking a unit step in the right (down) direction. Thus the walk starts with a step to the right, and ends at $(R_N,0)$ after $|S|$ steps. See the Figure 2 for the case of S as given by (1).

The lattice squares of $\mathbb{Z}^+\times\mathbb{Z}^+$ bounded by the right-down path determined by S constitute a polygonal region in the plane that we call the *staircase* corresponding to S. With this representation of the configuration-space, the exclusion-voter model can be viewed as a growth/depletion process on staircases. For instance, exclusion moves are particularly simple in this context, corresponding to adding or removing a square at a corner.

As well as $\mathcal{D}_0$, we introduce special notation for one more configuration. Set

$$\mathcal{D}_1 := \dots 11101000\dots, \tag{7}$$

the configuration with $N(\mathcal{D}_1)=1$ and vector representation $(1,1)$.

We now introduce notation for the changes in configuration brought about by voter and exclusion moves. Given the staircase of S, there are $2N+1$ 'corners' representing 10s and 01s alternately, of which $N+1$ are 10s and N are 01s. In the staircase representation, these corners have coordinates (R_i,T_{i+1}), $i\in\{0,\dots,N\}$ (for 10s) and (R_i,T_i), $i\in\{1,\dots,N\}$ (for 01s), where $R_0=T_{N+1}=0$. Enumerate the 10s left-to-right in the configuration S by $0,1,\dots,N$, and similarly the 01s by $1,\dots,N$.

For $j \in \{0, 1, \ldots, N\}$, let $v_j^{10\mapsto 00}(S)$, $v_j^{10\mapsto 11}(S)$ denote the configuration obtained from S by performing a voter move changing the jth 10 to 00, 11 respectively. Similarly for $j \in \{1, \ldots, N\}$ let $v_j^{01\mapsto 00}(S)$, $v_j^{01\mapsto 11}(S)$ denote the configuration obtained from the two possible voter moves at the jth 01. We use analogous notation for exclusion moves: $e_j^{10\mapsto 01}(S)$ ($j \in \{0, \ldots, N\}$), $e_j^{01\mapsto 10}(S)$ ($j \in \{1, \ldots, N\}$).

To conclude this section, we sketch the (elementary) proof of Proposition 1.

Proof of Proposition 1. It is not hard to see that $\mathcal{D}_0$ is an absorbing state for the pure voter model ($\beta = 1$) and for the left-moving totally asymmetric exclusion process ($\beta = 0, p = 1$), and hence also for the mixture model under $\mathbb{P}_{\beta,1}$ for any $\beta \in [0, 1]$.

To show that all states within $\mathcal{D}$ communicate, it suffices to show that $\mathbb{P}_{\beta,p}(\xi_{t+k} = S_1 \mid \xi_t = S_0) > 0$ for some $k \in \mathbb{N}$ for each of the following:

(i) $S_0 = \mathcal{D}_0$, $S_1 = \mathcal{D}_1$;

(ii) $S_0 = \mathcal{D}_1$, $S_1 = \mathcal{D}_0$;

(iii) any S_0 with $|S_0| \geq 2$ and some S_1 with $|S_1| = |S_0| + 1$;

(iv) any S_0 with $|S_0| \geq 3$ and some S_1 with $|S_1| \leq |S_0| - 1$;

(v) any S_0 with $|S_0| \geq 3$ and any S_1 where S_1 is identical to S_0 apart from in a single position $j \in \{2, 3, \ldots, |S_0| - 1\}$.

In other words, given that moves of types (i)–(v) can occur, it is possible (with positive probability) to step, in a finite number of moves, between any two configurations in $\mathcal{D}$ by first adjusting the length of the configuration via moves of types (i)–(iv) and then flipping the states in the interior of the configuration via moves of type (v). Similarly, to show that all states in $\mathcal{D} \setminus \{\mathcal{D}_0\}$ communicate, it suffices to show that all moves of types (iii)–(v) have positive probability.

It is not hard to see that voter moves can perform moves of types (ii), (iii) and (iv) in a single step (i.e., with $k = 1$). Similarly exclusion moves with $p < 1$ can perform moves of types (i) and (iii) in one step, while exclusion moves with $p > 0$ can perform moves of types (ii) and (iv), possibly needing multiple steps. We claim that moves of type (v) can be performed provided: (a) $\beta \in (0, 1)$; or (b) $\beta = 0$ and $p \in (0, 1)$.

In case (a), suppose we need to replace a 0 by a 1 in the interior of a given configuration. If $p < 1$, we may perform a voter move on the first 10 to the left of the position to be changed, and then, if necessary, perform successive $10 \mapsto 01$ exclusion moves to 'step' the 1 into the desired position. If $p > 0$, an analogous procedure works, starting from the first 01 to the *right*. On the other hand, if we need to replace a 1 by a 0, a similar argument applies.

In case (b) we cannot use voter moves, but both types of exclusion move are permitted, so we can 'bring in' any 0 (1) from outside the disordered region, rearrange as necessary, and 'take out' the excess 1 (0) to the other boundary.

It follows that moves of type (ii)–(v) are possible provided $\beta \neq 1$ and $(\beta, p) \notin \{(0, 0), (0, 1)\}$, and all (i)–(v) are possible if we additionally impose $p < 1$.

To complete the proof we need to demonstrate aperiodicity in the case where $\beta \neq 1$, $p < 1$ and $(\beta, p) \neq (0, 0)$, where all states communicate. Since $\beta \neq 1$, exclusion moves may occur. Moreover, every configuration other than $\mathcal{D}_0$ contains at least one pair of each type (01 and 10). Hence there is a positive probability that a configuration other than $\mathcal{D}_0$ remains unchanged at a given step (when a proposed exclusion move fails to occur). Thus, since all states communicate, we have aperiodicity. $\square$

4.3 Lyapunov function lemmas

Throughout this paper, Lyapunov-type functions will be primary tools. In this section we introduce some of our functions and give some preliminary results. Recall the definitions of R_i, T_i from (6). In [7], much use was made of the functions f_1, f_2 defined for $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ by

$$f_1(S) := \sum_{i=1}^{N} m_i R_i = \sum_{i=1}^{N} n_i T_i; \quad f_2(S) := \frac{1}{2}\left(\sum_{i=1}^{N} m_i R_i^2 + \sum_{i=1}^{N} n_i T_i^2\right),$$

and by $f_1(\mathcal{D}_0) = f_2(\mathcal{D}_0) = 0$. Note that with the diagrammatical representation described in Section 4.2, f_1 is the area of the staircase; e.g. for S given by (1), $f_1(S) = 162$.

In the present paper we introduce some more Lyapunov-type functions that will prove valuable: these include ρ^2 (see (43) below), ϕ_α for $\alpha > 0$ (see (22) below), and g which we define shortly. First we state some inequalities involving f_1 and f_2.

Lemma 3 *For any $S \in \mathcal{D}$, we have*

$$\frac{1}{2}|S| \leq f_1(S) \leq \frac{1}{4}|S|^2, \quad \text{and} \quad \frac{1}{4}|S|^2 \leq f_2(S) \leq \frac{1}{8}|S|^3; \tag{8}$$

$$f_2(S) \leq |S| f_1(S) \leq 2(f_1(S))^2. \tag{9}$$

Proof. The inequalities in (8) are in Lemma 4.1 of [7]. For (9), we have that for $S \in \mathcal{D}$

$$f_2(S) \leq \frac{1}{2}\left(\sum_{i=1}^{N} m_i R_i + \sum_{i=1}^{N} n_i T_i\right) \cdot (R_N + T_1) = f_1(S) \cdot |S|, \tag{10}$$

since, by (6), $R_i \leq R_N$ and $T_i \leq T_1$ for $1 \leq i \leq N$. Then from (10) and the first f_1 inequality in (8) we obtain (9). $\square$

The next lemma collects formulae for the expected increments of $f_1(\xi_t)$ and $f_2(\xi_t)$, obtained from (7.2), (5.3), and (6.3) in [7], that we will need. Note that (12) means that $f_2(\xi_t)$ is a martingale when $\beta = 1$.

Lemma 4 *Suppose $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ and $\beta, p \in [0, 1]$. Then*

$$\mathbb{E}_{\beta,p}[f_1(\xi_{t+1}) - f_1(\xi_t) \mid \xi_t = S] = (1-\beta)\frac{N(1-2p) + (1-p)}{2N+1} - \beta\frac{N}{2N+1}; \tag{11}$$

$$\mathbb{E}_{\beta,p}[f_2(\xi_{t+1}) - f_2(\xi_t) \mid \xi_t = S] = (1-\beta)\left(\frac{1}{2} + \frac{(1/2) - p}{2N+1} - \frac{2p-1}{2N+1}\sum_{i=1}^{N}(R_i + T_i)\right). \tag{12}$$

Next we define the function g, which captures most of f_1, in a sense made precise in Lemma 5 below. For $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, let $K = K(S)$ be the smallest member of $\{1, \ldots, N\}$ for which $R_K T_K = \max_{1 \leq k \leq N}\{R_k T_k\}$. Then for $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ set

$$X(S) := R_K; \quad Y(S) := T_K; \tag{13}$$

and put $X(\mathcal{D}_0) = Y(\mathcal{D}_0) = 0$. Then for $S \in \mathcal{D}$ we define

$$g(S) := X(S)Y(S) = \max_{1 \le k \le N} \{R_k T_k\}, \tag{14}$$

where $\max \emptyset := 0$. With the representation described in Section 4.2, g is the area of the largest rectangle that can be inscribed in the staircase.

Lemma 5 *For any $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$,*

$$f_1(S) \ge g(S) \ge \frac{f_1(S)}{1 + \log f_1(S)}. \tag{15}$$

Proof. We start with a geometrical argument that will yield the stated results via the staircase representation of configurations S. Define $r_a(x) := a/x$ for $a > 0$ and $x > 0$. For $a > 0$ and $b \ge 1$, let $R(a, b)$ denote the region defined by

$$R(a, b) := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le b, 0 \le y \le (a/x)\mathbf{1}_{\{x \ge 1\}} + a\mathbf{1}_{\{x<1\}}\}.$$

Then, with $|\cdot|$ denoting Lebesgue measure on $\mathbb{R}^2$,

$$|R(a, b)| = a + \int_1^b (a/x)\mathrm{d}x = a + a \log b.$$

Let $h : [0, \infty) \to [0, c]$ be a nonincreasing bounded function such that $h(x) = c$ for $0 \le x < 1$, $h(d) = 0$, and $h(x) \ge 1$ for $0 \le x < d$, where $c \ge 1$ and $d \ge 1$. Denote

$$M := M(h) := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le d, 0 \le y \le h(x)\}.$$

Let $a_0 := \sup\{a > 0 : \{r_a(x) : x > 0\} \cap M \ne \emptyset\}$, i.e. the greatest value of a for which a curve $r_a(x)$ intersects region M. Then let x_0 be such that $(x_0, r_{a_0}(x_0)) \in M$. Let $B(M)$ denote the rectangle with vertices $(0, 0)$, $(x_0, 0)$, $(0, r_{a_0}(x_0))$, and $(x_0, r_{a_0}(x_0))$; then $|B(M)| = x_0(a_0/x_0) = a_0$. Moreover, it is clear that $B(M) \subseteq M$ and $M \subseteq R(a_0, d)$, so

$$|B(M)| \le |M| \le |R(a_0, d)| = a_0(1 + \log d). \tag{16}$$

So, using the fact that $r_{a_0}(d) = a_0/d \ge 1$, we obtain from (16) that

$$1 \le \frac{|M|}{|B(M)|} \le 1 + \log d \le 1 + \log a_0 = 1 + \log |B(M)|. \tag{17}$$

Now we translate the above argument into a proof of the lemma. Fix a configuration $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ with block representation $(n_1, m_1, \ldots, n_N, m_N)$. For $x \ge 0$ define

$$j_S(x) := \max\{j \in \mathbb{Z}^+,\ j \le N : \sum_{i=1}^{j} n_i \le x\}; \quad \text{and} \quad h_S(x) := \sum_{i=j_S(x)+1}^{N} m_i,$$

where we interpret an empty sum as zero. Set $c_S = \sum_{i=1}^N m_i$ and $d_S = \sum_{i=1}^N n_i$. Then $h_S(x) = c_S$ when $0 \le x < 1$, since $n_1 \ge 1$. Also, $h_S(x) = 0$ for $x \ge d_S$ and $h_S(x) \ge m_N \ge 1$ for $0 \le x < d_S$. So h_S is a function of the form of h in the first paragraph of the present proof. In particular $|M(h_S)| = f_1(S)$ and $|B(M(h_S))| = g(S)$. Thus (17) implies (15). $\square$

5 Non-existence of passage-time moments

For $t \in \mathbb{Z}^+$, let $\mathcal{F}_t$ denote the σ-field generated by $(\xi_s; s \leq t)$. Recall the definitions of $X(S), Y(S)$ from (13). For convenience, we set $X_t := X(\xi_t)$ and $Y_t := Y(\xi_t)$ and consider the auxiliary $(\mathcal{F}_t)$-adapted process $(\tilde{\xi}_t)_{t\in\mathbb{Z}^+}$ defined by $\tilde{\xi}_t := (X_t, Y_t) = (X(\xi_t), Y(\xi_t))$; $\tilde{\xi}_t$ takes values in the quarter-lattice $\mathbb{Z}^+ \times \mathbb{Z}^+$, and $\xi_t = \mathcal{D}_0$ if and only if $\tilde{\xi}_t = (0, 0)$. Let $\sigma_{x,y}$ be the time for ξ_t to hit the ground configuration $\mathcal{D}_0$ (equivalently, the time taken for $\tilde{\xi}_t$ to hit the origin $(0, 0)$) given the $\mathcal{F}_0$-event $\{X(\xi_0) = x, Y(\xi_0) = y\}$. The crucial ingredient to the proof of non-existence of moments will be the following result.

Lemma 6 *Suppose $p \leq 1/2$, $\beta \in [0, 1]$. There exist $\delta > 0$, $\gamma > 0$ such that for all $x, y \in \mathbb{Z}^+$*

$$\mathbb{P}_{\beta,p}\left(\sigma_{x,y} \geq \delta(x^2 + y^2)\right) \geq \gamma, \tag{18}$$

and for all $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$

$$\mathbb{P}_{\beta,p}\left(\tau \geq \delta \frac{f_1(S)}{1 + \log f_1(S)} \mid \xi_0 = S\right) \geq \gamma. \tag{19}$$

Note that (19) is close to Conjecture 7.2 in [7]. The proof of Lemma 6 will be carried out in stages. The next result gives control over the size of the disordered region in the mixture process of voter model with *symmetric or recurrent* exclusion ($p \geq 1/2$).

Lemma 7 *Suppose $p \geq 1/2$ and $\beta \in [0, 1]$. Then for all $t \in \mathbb{N}$*

$$\mathbb{P}_{\beta,p}\left(\max_{0\leq s\leq t} |\xi_s| \leq 2\sqrt{10}t^{1/2}\right) \geq 0.95 - \frac{f_2(\xi_0)}{10t}.$$

Proof. For $p \geq 1/2$ and $\beta \in [0, 1]$, we have from (12) that $f_2(\xi_t)$ satisfies

$$\mathbb{E}_{\beta,p}[f_2(\xi_{t+1}) - f_2(\xi_t) \mid \xi_t = S] \leq \frac{1}{2},$$

for all $S \in \mathcal{D}$. Applying Lemma 1(i) to $f_2(\xi_t)$ with $r = 10t$ and $B = 1/2$, (5) implies that

$$\mathbb{P}_{\beta,p}\left(\max_{0\leq s\leq t} f_2(\xi_t) \leq 10t\right) \geq 1 - \frac{(t/2) + f_2(\xi_0)}{10t} = 0.95 - \frac{f_2(\xi_0)}{10t}.$$

Then using the fact that $|S| \leq 2(f_2(S))^{1/2}$ for any $S \in \mathcal{D}$ (by (8)), we obtain the result. $\square$

Suppose that $\xi_0 = S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ with corresponding $\tilde{\xi}_0 = (x_0, y_0) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e., $X(S_0) = x_0, Y(S_0) = y_0$. In order to enable us to identify positions within a configuration $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, enumerate the positions in the hybrid zone left to right as $1, 2, \ldots, |S|$.

Now we return to the voter plus transient ($p \leq 1/2$) exclusion model, and define an auxiliary *coloured* process as follows. Set $H := \sum_{i=1}^{K(S_0)}(n_i + m_i)$, recalling the definition of $K(S_0)$ from just above (13); then position H in S_0 is necessarily occupied by a 0 and position $H + 1$ by a 1. We *colour* the x_0 0s that occupy positions in $\{1, 2, \ldots, H\}$ and the y_0 1s that occupy positions in $\{H + 1, \ldots, |S_0|\}$. All other particles are uncoloured. Intuitively, coloured particles can be thought of as 'high energy'. Next we will define the evolution of the colouring corresponding to the process $(\xi_t)_{t\in\mathbb{Z}^+}$. We emphasize that the colouring is associated with the *particles* (i.e., 1s and 0s) rather than the *sites*.

The colour dynamics is as follows. Exclusion moves do not alter any colour, so that particles retain their colour-state after an exclusion move. Voter moves affect the colouring of particles only if the modified pair consists of exactly one coloured particle, in which case the colouring is changed as follows. In a pair 01 or 10 suppose that the 1 is coloured while the 0 is not; a voter move to pair 00 produces two uncoloured particles while a move to pair 11 produces two coloured particles. On the other hand, if in an unlike pair the 0 is coloured and the 1 not, a voter move to 00 produces two coloured particles and to 11 produces two uncoloured particles. We note the following facts about the dynamics:

(a) Uncoloured 1s remain to the left of any coloured 1s, and uncoloured 0s remain to the right of coloured 0s.

(b) A necessary condition to be in the ground configuration $\mathcal{D}_0$ is that the set of coloured particles consists only of a (possibly empty) block of coloured 1s at the left boundary of the hybrid zone and a (possibly empty) block of coloured 0s at the right boundary.

With $\xi_0 = S_0 \in \mathcal{D}$, for $t \in \mathbb{N}$ let ξ_t^* denote the configuration ξ_t with the associated colouring as determined by $(\xi_0, \ldots, \xi_t)$ according to the mechanism just described.

Let $\mathcal{F}_t^*$ denote the σ-field generated by $(\xi_s^*; s \leq t)$. Define the $\mathcal{F}_t^*$-measurable random variables ℓ_t and r_t as follows. Let ℓ_t be the position (measured from the left end of the hybrid zone) of the leftmost coloured 1 in ξ_t^* and r_t be the position of the rightmost coloured 0 in ξ_t^*; initially $r_0 + 1 = \ell_0$ by construction.

As the process evolves, coloured 1s may end up to the left of coloured 0s. We define an auxiliary process $(\zeta_t)_{t\in\mathbb{Z}^+}$ to keep track of such configurations. Informally, when $\ell_t < r_t$, ζ_t will be the portion of ξ_t between positions ℓ_t and r_t. More formally, we introduce a holding state $\mathcal{D}_0^*$ and set $\zeta_t = \mathcal{D}_0^*$ if $\ell_t \geq r_t$. If $\ell_t < r_t$, the configuration ξ_t^* induces a finite string of 0s and 1s obtained by extracting the segment of ξ_t^* between positions ℓ_t and r_t (inclusive); this string we call ζ_t. Then $(\zeta_t)_{t\in\mathbb{Z}^+}$ is an $(\mathcal{F}_t^*)$-adapted process with $\zeta_0 = \mathcal{D}_0^*$.

Note that, when it is not in state $\mathcal{D}_0^*$, ζ_t contains only *coloured* particles when colours are transposed from ξ_t^*. Now the idea is that when $p \leq 1/2$, ζ_t behaves like the mixture of voter and $p \geq 1/2$ exclusion, except that the presence of uncoloured particles in ξ_t^* causes it to 'slow down'; thus we aim for a version of Lemma 7 in this case. This is the next result.

Lemma 8 *Suppose $p \leq 1/2$ and $\beta \in [0,1]$. Then for all $t \in \mathbb{N}$*

$$\mathbb{P}_{\beta,p}\left(\max_{0\leq s\leq t} |\zeta_s| \leq 2\sqrt{10}t^{1/2}\right) \geq 0.95.$$

Proof. We compare the process $(\zeta_t)_{t\in\mathbb{Z}^+}$ to an independent copy $\xi' = (\xi'_t)_{t\in\mathbb{Z}^+}$ of the process ξ. We define $f_2^*(\zeta_t)$ analogously to $f_2(\xi_t)$, but counting only the (coloured) particles in region ζ_t, i.e. coloured 1s to the left of coloured 0s and coloured 0s to the right of coloured 1s. Suppose that initially we were to permit $\xi_0^* = \mathcal{D}'_0 := \ldots 000111 \ldots$ where all 0s and 1s are coloured, so that $\zeta_0 = \mathcal{D}_0^*$. Then by a simple reflection argument, the process $(\zeta_t)_{t\in\mathbb{Z}^+}$ embedded in $(\xi_t^*)_{t\in\mathbb{Z}^+}$ started from $\xi_0^* = \mathcal{D}'_0$ has the same distribution under $\mathbb{P}_{\beta,p}$ as the process $(\xi_t)_{t\in\mathbb{Z}^+}$ under $\mathbb{P}_{\beta,1-p}$ with initial state $\mathcal{D}_0$. So in particular Lemma 7 holds with ζ_t instead of ξ_t given the initial configuration $\mathcal{D}'_0$; then using the fact that $f_2^*(\zeta_0) = 0$ we obtain the claimed result in this case.

Now, the presence of uncoloured 1s to the left of coloured 1s or uncoloured 0s to the right of coloured 0s restricts the growth of $|\zeta_t|$; hence the claimed result also holds for any permissible initial configuration for ξ_0^* other than $\mathcal{D}_0'$. (One can argue rigorously by stochastic domination at this point.) $\square$

Proof of Lemma 6. We first prove the statement (18). Let $\chi_t := \chi(\xi_t^*)$ denote the number of coloured particles in ξ_t^*. Then $(\chi_t)_{t\in\mathbb{Z}^+}$ is $(\mathcal{F}_t^*)$-adapted and $\chi_0 = \chi(\xi_0^*) = x_0 + y_0$. Also, given $\chi_t = n$ for $n \in \mathbb{N}$ we have that $\chi_{t+1} = n$ unless a voter move is performed on a pair with exactly one particle coloured, in which case χ_{t+1} takes values $n-1, n+1$ with equal probability. Also if $\chi_t = 0$ then $\chi_{t+1} = 0$ as well. Thus χ_t is a nonnegative $(\mathcal{F}_t^*)$-martingale with uniformly bounded jumps. It follows from Doob's submartingale inequality applied to the nonnegative submartingale $(\chi_t - (x_0+y_0))^2$, using the fact that $\mathbb{E}[(\chi_t - (x_0+y_0))^2] \le t$ by the orthogonality of martingale increments, that for any $z > 0$

$$\mathbb{P}_{\beta,p}\left(\max_{0\le s\le t} |\chi_s - (x_0+y_0)|^2 \ge z\right) \le t/z.$$

In particular, taking $z = 100t$ this implies that for any $t \in \mathbb{Z}^+$

$$\mathbb{P}_{\beta,p}\left(\min_{0\le s\le t} \chi_s \ge (x_0+y_0) - 10t^{1/2}\right) \ge 0.99.$$

Taking $t = \delta^2(x_0^2+y_0^2)$ for some $\delta > 0$, combining the last display with Lemma 8, we have that with probability at least 0.94 the two events

$$\left\{\min_{0\le s\le t} \chi_s \ge (x_0+y_0) - 10\delta(x_0^2+y_0^2)^{1/2} \ge (1-10\delta)(x_0+y_0)\right\}$$
$$\text{and} \quad \left\{\max_{0\le s\le t} |\zeta_s| \le 2\sqrt{10}\delta(x_0^2+y_0^2)^{1/2} \le 2\sqrt{10}\delta(x_0+y_0)\right\}$$

both occur (noting that $(x_0^2+y_0^2)^{1/2} \le (x_0+y_0)$). Choose δ small, say $\delta = 0.01$. Then with probability at least 0.94 the total number χ_s of coloured particles up to time t remains greater than $0.9(x_0+y_0)$ while the central overlap region ζ_s of coloured particles between the leftmost coloured 1 and the rightmost coloured 0 remains shorter than $0.1(x_0+y_0)$. Hence there must remain at least one coloured 0 to the left of any coloured 1 or one coloured 1 to the right of any coloured 0. By observation (b) above, this excludes the possibility of $\xi_s = \mathcal{D}_0$ for any $s \le t$, where $t = \delta^2(x_0^2+y_0^2)$. Thus we obtain (18).

To derive (19), we use the fact that $x_0^2 + y_0^2 \ge 2x_0y_0 = 2g(\xi_0)$ and now use (15). $\square$

Now we are nearly ready to complete the proofs of Theorems 2 and 3. The proofs proceed in a similar way to the proof of Theorem 6.1 in [7].

Proof of Theorem 2. Suppose $p \le 1/2$. Take $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$. Suppose, for the purpose of deriving a contradiction, that $\mathbb{E}_{\beta,p}[\tau^{1+\varepsilon} \mid \xi_0 = S_0] < \infty$ for some $\varepsilon > 0$. Let $\xi' = (\xi_t')_{t\in\mathbb{Z}^+}$ be an independent copy of ξ and τ' be the corresponding independent copy of τ. For any $t \in \mathbb{Z}^+$, using the Markov property we obtain

$$\mathbb{E}_{\beta,p}[\tau^{1+\varepsilon} \mid \xi_0 = S_0] \ge \mathbb{E}_{\beta,p}\left[\mathbb{E}_{\beta,p}\left[(t+\tau')^{1+\varepsilon} \mid \xi_0' = \xi_t\right] \mathbf{1}_{\{\tau\ge t\}} \mid \xi_0 = S_0\right]. \qquad (20)$$

For the inner expectation in the expression on the right-hand side of (20), we have by (19) that there exist $\delta > 0$, $\gamma > 0$ such that for any $t \in \mathbb{Z}^+$

$$\mathbb{E}_{\beta,p}\left[(t+\tau')^{1+\varepsilon} \mid \xi_0' = \xi_t\right] \geq \gamma \left(\delta \frac{f_1(\xi_t)}{1+\log f_1(\xi_t)}\right)^{1+\varepsilon}.$$

Since for any $\varepsilon > 0$ $x^\varepsilon > 1 + \log x$ for all x sufficiently large, there exists $\gamma' > 0$ for which

$$\mathbb{E}_{\beta,p}\left[(t+\tau')^{1+\varepsilon} \mid \xi_0' = \xi_t\right] \geq \gamma' \left(f_1(\xi_t)^{1-(\varepsilon/2)}\right)^{1+\varepsilon}, \tag{21}$$

for any $t \in \mathbb{Z}^+$. It follows from (20) with (21) that for some $\varepsilon' \in (0, \varepsilon)$ and some $C \in (0, \infty)$

$$\mathbb{E}_{\beta,p}[\tau^{1+\varepsilon} \mid \xi_0 = S_0] \geq C\mathbb{E}_{\beta,p}[(f_1(\xi_t))^{1+\varepsilon'}\mathbf{1}_{\{\tau \geq t\}} \mid \xi_0 = S_0] = C\mathbb{E}_{\beta,p}[(f_1(\xi_{t\wedge\tau}))^{1+\varepsilon'} \mid \xi_0 = S_0],$$

for any $t \in \mathbb{Z}^+$, using the fact that a.s. $f_1(\xi_\tau) = f_1(\mathcal{D}_0) = 0$. That is, given $\xi_0 = S_0$, $(f_1(\xi_{t\wedge\tau}))^{1+\varepsilon'}$ is uniformly bounded in L^1.

Hence the assumption that $\mathbb{E}_{\beta,p}[\tau^{1+\varepsilon} \mid \xi_0 = S_0] < \infty$ implies that on $\xi_0 = S_0$ the process $f_1(\xi_{t\wedge\tau})$ is uniformly integrable, and trivially that $\tau < \infty$ a.s.; thus as $t \to \infty$, $\mathbb{E}_{\beta,p}[f_1(\xi_{t\wedge\tau}) \mid \xi_0 = S_0] \to \mathbb{E}_{\beta,p}[f_1(\xi_\tau) \mid \xi_0 = S_0] = f_1(\mathcal{D}_0) = 0$. However, for $p \leq 1/2$ and $\beta \leq (1-2p)/(2-2p)$, it follows from (11) that any $t \in \mathbb{Z}^+$ and any $S \in \mathcal{D}$

$$\mathbb{E}_{\beta,p}[f_1(\xi_{t+1}) - f_1(\xi_t) \mid \xi_t = S] \geq 0.$$

Then by the submartingale property we have that for all $t \in \mathbb{Z}^+$, $\mathbb{E}_{\beta,p}[f_1(\xi_{t\wedge\tau}) \mid \xi_0 = S_0] \geq f_1(S_0) > 0$. Thus we have the desired contradiction. $\square$

Proof of Theorem 3. Suppose that $S_0 \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, and, for a contradiction, that $\mathbb{E}_{\beta,p}[\tau^{2+\varepsilon} \mid \xi_0 = S_0] < \infty$ for some $\varepsilon > 0$. Then, for any $t \in \mathbb{Z}^+$, similarly to the proof of Theorem 2,

$$\mathbb{E}_{\beta,p}[\tau^{2+\varepsilon} \mid \xi_0 = S_0] \geq \mathbb{E}_{\beta,p}\left[\mathbb{E}_{\beta,p}\left[(t+\tau')^{2+\varepsilon} \mid \xi_0' = \xi_t\right]\mathbf{1}_{\{\tau \geq t\}} \mid \xi_0 = S_0\right].$$

Hence for $p \leq 1/2$, using (19), there exist $\gamma, \delta, \varepsilon', \varepsilon'' > 0$ such that

$$\begin{aligned}\mathbb{E}_{\beta,p}[\tau^{2+\varepsilon} \mid \xi_0 = S_0] &\geq \gamma\mathbb{E}_{\beta,p}\left[(t+\delta(f_1(\xi_t))^{1-(\varepsilon/3)})^{2+\varepsilon}\mathbf{1}_{\{\tau \geq t\}} \mid \xi_0 = S_0\right] \\ &\geq C\mathbb{E}_{\beta,p}[(f_1(\xi_{t\wedge\tau}))^{2+\varepsilon'} \mid \xi_0 = S_0] \geq C\mathbb{E}_{\beta,p}[(f_2(\xi_{t\wedge\tau}))^{1+\varepsilon''} \mid \xi_0 = S_0],\end{aligned}$$

using (9) for the last inequality. Hence the process $f_2(\xi_{t\wedge\tau})$ is uniformly integrable, and thus as $t \to \infty$, $\mathbb{E}_{\beta,p}[f_2(\xi_{t\wedge\tau}) \mid \xi_0 = S_0] \to \mathbb{E}_{\beta,p}[f_2(\xi_\tau) \mid \xi_0 = S_0] = f_2(\mathcal{D}_0) = 0$.

However, for $p \leq 1/2$ and $\beta \in [0,1]$, for all $S \in \mathcal{D}$ and all $t \in \mathbb{Z}^+$ it follows from (12) that

$$\mathbb{E}_{\beta,p}[f_2(\xi_{t+1}) - f_2(\xi_t) \mid \xi_t = S] \geq 0.$$

Hence for all $t \in \mathbb{Z}^+$, $\mathbb{E}_{\beta,p}[f_2(\xi_{t\wedge\tau}) \mid \xi_0 = S_0] \geq f_2(S_0) > 0$, giving a contradiction. $\square$

6 Recurrence

We consider a new Lyapunov-type function that generalizes f_1. For $\alpha \geq 0$, set $\phi_\alpha(\mathcal{D}_0) := 0$ and for $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ set

$$\phi_\alpha(S) := \sum_{i=1}^{N} \sum_{j=R_{i-1}+1}^{R_i} \sum_{k=1}^{T_i} \frac{1}{(j+k)^\alpha} = \sum_{i=1}^{N} \sum_{j=T_{i+1}+1}^{T_i} \sum_{k=1}^{R_i} \frac{1}{(j+k)^\alpha}; \tag{22}$$

here and throughout this section we use the conventions $R_0 := 0, R_{N+1} := R_N, T_0 := T_1, T_{N+1} := 0$. In particular it follows from (22) that when $\alpha = 0$, $\phi_0(S) = \sum_{i=1}^N n_i T_i = \sum_{i=1}^N m_i R_i = f_1(S)$. For convenience we introduce the notation

$$a_i(j) := (T_j + R_j + i)^{-\alpha}, \quad \text{and} \quad b_i(j) := (T_{j+1} + R_j + i)^{-\alpha}.$$

The next lemma gives an expression for the expected increments of ϕ_α.

Lemma 9 *Let $\beta \in [0,1]$ and $p \in [0,1]$. Then for any $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ and any $t \in \mathbb{Z}^+$*

$$\begin{aligned} \mathbb{E}_{\beta,p}[\phi_\alpha(\xi_{t+1}) - \phi_\alpha(\xi_t) \mid \xi_t = S] &= \frac{1-\beta}{2N+1}\left\{-p\sum_{j=1}^N a_0(j) + (1-p)\sum_{j=0}^N b_2(j)\right\} \\ &+ \frac{\beta}{2N+1}\left\{N\sum_{j=1}^N (a_1(j) - b_1(j-1)) - \frac{1}{2}\sum_{j=2}^N a_1(j) - \frac{N+1}{2}b_1(N)\right\}. \end{aligned} \tag{23}$$

Proof. Recalling the notation of Section 4.2, write

$$\begin{aligned} D_j^{\mathrm{v},10}(S) &:= \phi_\alpha(v_j^{10\mapsto 00}(S)) + \phi_\alpha(v_j^{10\mapsto 11}(S)) - 2\phi_\alpha(S) \quad (j \in \{0,\dots,N\}); \\ D_j^{\mathrm{v},01}(S) &:= \phi_\alpha(v_j^{01\mapsto 00}(S)) + \phi_\alpha(v_j^{01\mapsto 11}(S)) - 2\phi_\alpha(S) \quad (j \in \{1,\dots,N\}); \\ D_j^{\mathrm{e},10}(S) &:= \phi_\alpha(e_j^{10\mapsto 01}(S)) - \phi_\alpha(S) \quad (j \in \{0,\dots,N\}); \\ D_j^{\mathrm{e},01}(S) &:= \phi_\alpha(e_j^{01\mapsto 10}(S)) - \phi_\alpha(S) \quad (j \in \{1,\dots,N\}). \end{aligned}$$

Summing over all possible moves we have that

$$\begin{aligned} \mathbb{E}_{\beta,p}[\phi_\alpha(\xi_{t+1}) - \phi_\alpha(\xi_t) \mid \xi_t = S] &= \frac{\beta}{2N+1}\left\{\frac{1}{2}\sum_{j=1}^N D_j^{\mathrm{v},01}(S) + \frac{1}{2}\sum_{j=0}^N D_j^{\mathrm{v},10}(S)\right\} \\ &+ \frac{1-\beta}{2N+1}\left\{-p\sum_{j=1}^N D_j^{\mathrm{e},01}(S) + (1-p)\sum_{j=0}^N D_j^{\mathrm{e},10}(S)\right\}. \end{aligned} \tag{24}$$

Now we calculate expressions for the terms in (24). The reader might find it helpful to refer to a picture such as Figure 2 in Section 4.2 here. We have that for $j \in \{1,\dots,N\}$

$$D_j^{\mathrm{v},01}(S) = \sum_{i=1}^N (a_1(i) - b_1(i-1)) - a_1(j) - a_1(j+1).$$

Also for $j \in \{0,1,\dots,N\}$ we have that $D_j^{\mathrm{v},10}(S)$ is given by

$$\sum_{i=1}^N \big(a_1(i) - \mathbf{1}_{\{i\le j\}} b_1(i-1) - \mathbf{1}_{\{i>j\}} b_1(i)\big) = \sum_{i=1}^N (a_1(i) - b_1(i-1)) + b_1(j) - b_1(N).$$

Taking the computations for $D_j^{\mathrm{v},01}(S)$, $D_j^{\mathrm{v},10}(S)$ and summing we have

$$\frac{1}{2}\sum_{j=1}^N D_j^{\mathrm{v},01}(S) + \frac{1}{2}\sum_{j=0}^N D_j^{\mathrm{v},10}(S)$$

$$= \frac{2N+1}{2}\sum_{j=1}^{N}(a_1(j)-b_1(j-1)) - \frac{1}{2}\sum_{j=1}^{N}(a_1(j)+a_1(j+1)) + \frac{1}{2}\sum_{j=0}^{N} b_1(j) - \frac{N+1}{2}b_1(N)$$
$$= \frac{2N+1}{2}\sum_{j=1}^{N}(a_1(j)-b_1(j-1)) - \sum_{j=1}^{N} a_1(j) + \frac{1}{2}a_1(1) + \frac{1}{2}\sum_{j=1}^{N} b_1(j-1) - \frac{N+1}{2}b_1(N)$$
$$= N\sum_{j=1}^{N}(a_1(j)-b_1(j-1)) - \frac{1}{2}\sum_{j=2}^{N} a_1(j) - \frac{N+1}{2}b_1(N).$$

For the (simpler) exclusion moves, we obtain $D_j^{\mathrm{e},01}(S) = -a_0(j)$ and $D_j^{\mathrm{e},10}(S) = b_2(j)$. Then combining all the computations, from (24) we obtain (23). □

For the rest of this section we will be interested in the properties of ϕ_1.

Lemma 10 *For any $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$, $\phi_1(S) \geq \log(|S|/4)$.*

Proof. Suppose $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$. From (22) we have that

$$\phi_1(S) \geq \sum_{i=1}^{N} \sum_{j=R_{i-1}+1}^{R_i} \frac{1}{1+j} = \sum_{j=1}^{R_N} \frac{1}{j+1} \geq \int_1^{R_N} \frac{\mathrm{d}x}{1+x} \geq \log(R_N/2),$$

using monotonicity for the second inequality. Similarly (22) gives $\phi_1(S) \geq \log(T_1/2)$. Thus $\phi_1(S) \geq \log(\max\{R_N, T_1\}/2)$, which yields the result. □

The following lemma is the key to this section.

Lemma 11 *Suppose $\beta \geq 4/7$ and $p \in [0,1]$. Then for any $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$,*

$$\mathbb{E}_{\beta,p}[\phi_1(\xi_{t+1}) - \phi_1(\xi_t) \mid \xi_t = S] \leq 0.$$

Proof. For ease of notation during this proof, set $\Delta(S) := \mathbb{E}_{\beta,p}[\phi_1(\xi_{t+1}) - \phi_1(\xi_t) \mid \xi_t = S]$. It is clear from (23) that $\Delta(S)$ is non-increasing in p, and so it suffices to consider the case $p = 0$. Then (23) implies that in this case $\Delta(S)$ is given by

$$\frac{\beta}{2N+1}\left\{N\sum_{j=1}^{N}(a_1(j)-b_1(j-1)) - \frac{1}{2}\sum_{j=2}^{N} a_1(j) - \frac{N+1}{2}b_1(N)\right\} + \frac{1-\beta}{2N+1}\sum_{j=0}^{N} b_2(j).$$

We rewrite this last expression by setting $\gamma := (1-\beta)/\beta \in [0,\infty)$ to obtain

$$\frac{2N+1}{\beta}\Delta(S) = N\sum_{j=1}^{N}(a_1(j)-b_1(j-1)) - \frac{1}{2}\sum_{j=2}^{N} a_1(j) - \frac{1}{2}\frac{N+1}{R_N+1} + \gamma\sum_{j=1}^{N+1} b_2(j-1). \tag{25}$$

We need to show that the right-hand side of (25) is non-positive. Since this quantity is nondecreasing in γ, it suffices to consider the case $\gamma = 3/4$, corresponding to $\beta = 4/7$. Set

$$\tilde{\Delta}(S) := N\sum_{j=1}^{N}(a_1(j)-b_1(j-1)) - \frac{1}{2}\sum_{j=2}^{N} a_1(j) - \frac{1}{2}\frac{N+1}{R_N+1} + \frac{3}{4}\sum_{j=1}^{N+1} b_1(j-1),$$

so that, from (25), $\Delta(S) \leq \frac{\beta}{2N+1}\tilde{\Delta}(S)$ since $b_1(j) \geq b_2(j)$.

Write $A_N := 1 + m_1 + m_2 + \cdots + m_N$, $D_0 := 0$ and, for $i \in \{1, \ldots, N\}$, $D_i := (n_1 - m_1) + \cdots + (n_i - m_i)$, so that $R_{j-1} + T_j + 1 = A_N + D_{j-1}$. Then we have that

$$\tilde{\Delta}(S) = N \sum_{j=1}^{N} \left(\frac{1}{A_N + D_{j-1} + n_j} - \frac{1}{A_N + D_{j-1}} \right) - \sum_{j=2}^{N} \frac{1/2}{A_N + D_{j-1} + n_j}$$
$$- \frac{(N+1)/2}{A_N + D_N} + \frac{3}{4} \sum_{j=1}^{N+1} \frac{1}{A_N + D_{j-1}} = \frac{1/2}{A_N + n_1} + H_N(S), \tag{26}$$

where we have introduced the notation, for $k \in \{1, \ldots, N\}$,

$$H_k(S) := \sum_{j=1}^{k} \left(\frac{N - 1/2}{A_N + D_{j-1} + n_j} - \frac{N - 3/4}{A_N + D_{j-1}} \right) - \frac{(N + k - 1)/4}{A_N + D_k}.$$

We now claim that if $N \geq 2$, for any $k \in \{2, \ldots, N\}$,

$$H_k(S) \leq H_{k-1}(S). \tag{27}$$

Then we have from (26) and (27) that for $N \geq 2$

$$\tilde{\Delta}(S) = \frac{1/2}{A_N + n_1} + H_N(S) \leq \frac{1/2}{A_N + n_1} + H_1(S) = \frac{N}{A_N + n_1} - \frac{N - 3/4}{A_N} - \frac{N/4}{A_N + n_1 - m_1}$$
$$\leq \frac{N}{A_N + n_1} - \frac{N - 3/4}{A_N + n_1} - \frac{N/4}{A_N + n_1} = \frac{(3 - N)/4}{A_N + n_1}.$$

Thus $\tilde{\Delta}(S) \leq 0$, and hence $\Delta(S) \leq 0$ also, for all $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$ with $N(S) \geq 3$.

Let us now verify the claim (27). We have that for $k \in \{2, \ldots, N\}$

$$H_k(S) = \sum_{j=1}^{k-1} \left(\frac{N - 1/2}{A_N + D_{j-1} + n_j} - \frac{N - 3/4}{A_N + D_{j-1}} \right) + \frac{N - (1/2)}{A_N + D_{k-1} + n_k}$$
$$- \frac{N - (3/4)}{A_N + D_{k-1}} - \frac{(N + k - 1)/4}{A_N + D_k}$$
$$= \sum_{j=1}^{k-1} \left(\frac{N - 1/2}{A_N + D_{j-1} + n_j} - \frac{N - 3/4}{A_N + D_{j-1}} \right) + \left[\frac{(3N - k - 1)/4}{A_N + D_{k-1} + n_k} - \frac{N - (3/4)}{A_N + D_{k-1}} \right]$$
$$+ \left[\frac{(N + k - 1)/4}{A_N + D_{k-1} + n_k} - \frac{(N + k - 1)/4}{A_N + D_k} \right],$$

where we have split the term with denominator $A_N + D_{k-1} + n_k$ into two parts. Note that for all j we have $A_N + D_{j-1} + n_j \geq A_N + D_{j-1}$ and also $A_N + D_{j-1} + n_j = A_N + D_j + m_j \geq A_N + D_j$. Therefore, applying these inequalities separately to the two terms in square brackets in the last display, we verify the claim (27) since

$$H_k(S) \leq \sum_{j=1}^{k-1} \left(\frac{N - 1/2}{A_N + D_{j-1} + n_j} - \frac{N - 3/4}{A_N + D_{j-1}} \right) + \left[\frac{(-N - k + 2)/4}{A_N + D_{k-1}} \right] = H_{k-1}(S).$$

To complete the proof, we show that $\Delta(S) \leq 0$ for $N(S) \in \{1,2\}$ also. For $N = 1$, writing the right-hand side of the $\beta = 4/7$ case of (25) over a common denominator we have

$$\frac{21}{4}\Delta(S) = -\frac{n_1 m_1(n_1 - m_1)^2 + 13(n_1 + m_1) + 4(1 + n_1 m_1) + 14(n_1^2 + m_1^2) + 5(n_1^3 + m_1^3)}{4(1 + m_1 + n_1)(1 + m_1)(1 + n_1)(2 + m_1)(2 + n_1)},$$

which is negative. Finally, for $N = 2$, from (25) and some tedious algebra we obtain $35\Delta(S)/4 = -Q/R$ where $R = 4(m_1 + m_2 + n_1 + 1)(m_1 + m_2 + 1)(m_2 + n_1 + n_2 + 1)(m_2 + n_1 + 1)(n_1 + n_2 + 1)(m_1 + m_2 + 2)(m_2 + n_1 + 2)(n_1 + n_2 + 2)$ and

$$Q = m_1 n_1^4(2m_1^2 + 5n_1^2 - m_1 n_1) + n_2 m_2^4(2n_2^2 + 5m_2^2 - n_2 m_2) + 244 \text{ positive terms}$$

as can be readily checked in `Maple`, for instance. Since $2x^2 + 5y^2 - xy$ is always nonnegative, we conclude that $\Delta(S) \leq 0$ in this last case also. □

Proof of Theorem 4. Lemma 11 shows that for $\beta \geq 4/7$, $(\phi_1(\xi_t))_{t \in \mathbb{Z}^+}$ is a supermartingale on $\xi_t \in \mathcal{D} \setminus \{\mathcal{D}_0\}$. Since, by Lemma 10, $\phi_1(S) \to \infty$ as $|S| \to \infty$, we can use Theorem 2.2.1 of [11] to complete the proof of the theorem. □

7 Existence of passage-time moments

Our main tool in this section will be Lemma 2 applied with the Lyapunov function f_2. Our first result is a bound on the expected increments of f_2.

Lemma 12 *Suppose $\beta \in [0,1)$ and $p > 1/2$. Then there exists $C \in (0,\infty)$ such that for all but finitely many $S \in \mathcal{D}$*

$$\mathbb{E}_{\beta,p}[f_2(\xi_{t+1}) - f_2(\xi_t) \mid \xi_t = S] \leq -C(f_2(S))^{1/6}.$$

Proof. It follows directly from (6) that $R_N + T_1 = |S|$ and $R_i + T_i \geq N$, so that

$$\sum_{i=1}^{N}(R_i + T_i) \geq \max\{|S|, N^2\} \geq N|S|^{1/2}. \tag{28}$$

We see from (12) with (28) that for $p > 1/2$, $\mathbb{E}_{\beta,p}[f_2(\xi_{t+1}) - f_2(\xi_t) \mid \xi_t = S]$ is at most

$$\frac{1-\beta}{2} - (1-\beta)(2p-1)\frac{N|S|^{1/2}}{2N+1} \leq \frac{1-\beta}{2} - \frac{1}{3}(1-\beta)(2p-1)|S|^{1/2},$$

since $N \geq 1$. The result follows since $|S|^{1/2} \geq f_2(S)^{1/6}$ from (8). □

Proof of Theorem 6. The $\beta = 1$ case of the theorem follows from Theorem 1(ii). Now suppose $\beta \in [0,1)$. Applying Lemma 2 with $X_t = f_2(\xi_t)$ and using Lemma 12 shows that the hitting time of a finite subset of $\mathcal{D}$ has finite $(6/5)$-th moment. Since $\mathcal{D}_0$ is accessible from any state, it follows that τ also has finite $(6/5)$-th moment. □

Remark. The exponent 1/6 in Lemma 12 may not be best possible. However, (12) applied to the configuration $n_1 = n_2 = \cdots = n_{N-1} = 1, n_N = N^2$ and $m_1 = N^2, m_2 = \cdots = m_N = 1$ shows that one cannot increase the exponent to more than 1/4 in general. Hence the method used in this section seems unable to prove existence of moments greater than 4/3; see the remark just after the statement of Theorem 6.

To prove Theorem 5 we will again apply Lemma 2, but this time we will take $X_t = f_2(\xi_t)^M$ for arbitrary $M \in [1, \infty)$. To study the increments of this process we recall some facts about f_2 under exclusion moves; compare (5.1) and (5.2) in [7]. We have that

$$f_2(e_j^{10\mapsto 01}(S)) = f_2(S) + 1 + R_j + T_{j+1}, \quad (j \in \{0, \dots, N\}); \tag{29}$$
$$f_2(e_j^{01\mapsto 10}(S)) = f_2(S) + 1 - R_j - T_j, \quad (j \in \{1, \dots, N\}), \tag{30}$$

where $R_0 := 0$ and $T_{N+1} := 0$. Now we will prove the following lemma.

Lemma 13 *Suppose $\beta = 0$ and $p > 1/2$. Let $M \in (0, \infty)$. Then there exists $C \in (0, \infty)$ such that for all but finitely many $S \in \mathcal{D}$*

$$\mathbb{E}_{0,p}[f_2(\xi_{t+1})^M - f_2(\xi_t)^M \mid \xi_t = S] \leq -C f_2(S)^{M-(5/6)}.$$

Proof. In this proof we write $\Delta_2(S) := \mathbb{E}_{0,p}[f_2(\xi_{t+1})^M - f_2(\xi_t)^M \mid \xi_t = S]$ which we calculate by summing over all the possible exclusion moves. The $e_j^{10\mapsto 01}$ transition has probability $(1-p)/(2N+1)$ and changes $f_2(S)^M$ by

$$(f_2(S) + 1 + R_j + T_{j+1})^M - f_2(S)^M = f_2(S)^M \left[\left(1 + \frac{1 + R_j + T_{j+1}}{f_2(S)} \right)^M - 1 \right].$$

Since $R_j + T_{j+1} \leq |S| = O(f_2(S)^{1/2})$ for any j, by (8), Taylor's theorem yields

$$(f_2(S) + 1 + R_j + T_{j+1})^M - f_2(S)^M = f_2(S)^M \left[M \frac{1 + R_j + T_{j+1}}{f_2(S)} + O(f_2(S)^{-1}) \right].$$

Proceeding similarly for the $e_j^{01\mapsto 10}$ transitions and summing we obtain

$$\Delta_2(S) = \frac{M}{2N+1} f_2(S)^{M-1} \left[N + (1-p) + (1-2p) \sum_{j=1}^{N} (R_j + T_j) \right] + O(f_2(S)^{M-1}), \tag{31}$$

where the implicit constant in $O(\cdot)$ does not depend on S. Then from (31) with (28) and (8) we obtain that for some $C_1, C_2 \in (0, \infty)$

$$\Delta_2(S) \leq C_1 f_2(S)^{M-1} - C_2 f_2(S)^{M-(5/6)}.$$

This yields the result. □

Proof of Theorem 5. Take $X_t = f_2(\xi_t)^M$ for some $M \geq 1$. Then we have from Lemma 13 that $\mathbb{E}[X_{t+1} - X_t \mid \xi_t = S] \leq -C X_t^{1-\frac{5}{6M}}$ for all but finitely many S. Thus we can apply Lemma 2 to obtain $\mathbb{E}[\tau^{6M/5}] < \infty$. Since $M \geq 1$ was arbitrary, the theorem follows. □

8 Size of the hybrid zone

We now prove the almost-sure bounds on the rate of growth of $|\xi_t|$ stated in Section 2.

Lemma 14 *Let $\beta \in [0,1]$, $p \in [0,1]$. For any $\varepsilon > 0$, $\mathbb{P}_{\beta,p}$-a.s., for all but finitely many t,*

$$\max_{0 \le s \le t} f_1(\xi_s) \le t(\log t)^{1+\varepsilon}.$$

Proof. From (11) we have that for any $S \in \mathcal{D}$

$$\mathbb{E}_{\beta,p}[f_1(\xi_{t+1}) - f_1(\xi_t) \mid \xi_t = S] \le \frac{N(S)+1}{2N(S)+1} \le 1.$$

Then we can apply Lemma 1(ii) with $X_t = f_1(\xi_t)$ to obtain the result. $\square$

Proof of Theorem 7. Lemma 14 with the simple inequality $f_1(\xi_t) \ge N(\xi_t)^2/2$ implies the $p < 1/2$ case of (2). By (12) we have that for $p \ge 1/2$ and all $S \in \mathcal{D}$

$$\mathbb{E}_{\beta,p}[f_2(\xi_{t+1}) - f_2(\xi_t) \mid \xi_t = S] \le \frac{1-\beta}{2}.$$

Hence Lemma 1(ii) with $X_t = f_2(\xi_t)$ yields, for any $\varepsilon > 0$, $\mathbb{P}_{\beta,p}$-a.s.,

$$\max_{0 \le s \le t} f_2(\xi_s) \le t(\log t)^{1+\varepsilon}, \tag{32}$$

for all but finitely many t. Then (3) follows from (32) with (8), and the $p \ge 1/2$ case of (2) follows from (32) with the simple inequality $f_2(\xi_t) \ge (N(\xi_t))^3/3$ (obtained by replacing each m_i and n_i by 1 in the definition of f_2). $\square$

For the remainder of this section, we concentrate on the pure exclusion process, i.e. when $\beta = 0$. Again the Lyapunov function f_1 will be a primary tool here; the next result describes its behaviour in this case. We use the abbreviation $N_t := N(\xi_t)$.

Lemma 15 *Let $\beta = 0$, $p \in [0,1]$. Then $f_1(\xi_t)$ has transition probabilities $p_j = \mathbb{P}_{0,p}(f_1(\xi_{t+1}) - f_1(\xi_t) = j \mid \mathcal{F}_t)$ for jumps $j \in \{-1,0,+1\}$ where $p_{-1} + p_0 + p_1 = 1$ and*

$$p_{-1} = p\frac{N_t}{2N_t+1} \le \frac{p}{2}, \quad p_0 = \frac{N_t + p}{2N_t+1}, \quad p_1 = (1-p)\frac{N_t+1}{2N_t+1} \ge \frac{1-p}{2}. \tag{33}$$

Hence for all $t \in \mathbb{Z}^+$,

$$f_1(\xi_t) \le f_1(\xi_0) + t. \tag{34}$$

Moreover, when $p < 1/2$, for any $c \in (0,(1/2)-p)$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many t,

$$f_1(\xi_t) \ge ct. \tag{35}$$

Proof. (33) follows from equations (5.5) and (5.6) in [7]. Then (34) is immediate. From (33), we have that ξ_t stochastically dominates $\xi_0 + \sum_{s=1}^t W_s$ where $W_1, W_2, \ldots$ are i.i.d. random variables taking values $+1$, 0, -1 with probabilities $q/2$, $1/2$, $p/2$ respectively. Hence the SLLN and the fact that $\mathbb{E}[W_1] = (1/2) - p$ yields (35) for $p < 1/2$. $\square$

Corollary 2 *Suppose $\beta = 0$ and $p < 1/2$. Then there exists $c'(p) > 0$ such that for any $c \in (0, c'(p))$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many t,*

$$|\xi_t| \ge ct^{1/2}. \tag{36}$$

Suppose $\beta = 0$. Then there exists $C \in (0,\infty)$ such that for any $p \in [0,1]$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$

$$N(\xi_t) \le Ct^{1/2}. \tag{37}$$

Proof. The bound (36) follows from (35) together with (8); (37) follows from (34) with the simple inequality $f_1(\xi_t) \geq (N(\xi_t))^2/2$. □

The next two lemmas give some properties of the process $(|\xi_t|)_{t\in\mathbb{Z}^+}$. Recall the definition of configuration $\mathcal{D}_1$ from (7).

Lemma 16 *Suppose $\beta = 0$ and $p \in [0,1]$. For any $t \in \mathbb{Z}^+$, we have that*

$$\mathbb{P}_{0,p}(|\xi_{t+1}| = 2 \mid \xi_t = \mathcal{D}_0) = 1 - \mathbb{P}_{0,p}(|\xi_{t+1}| = 0 \mid \xi_t = \mathcal{D}_0) = 1 - p; \quad (38)$$

$$\text{and} \quad \mathbb{P}_{0,p}(|\xi_{t+1}| = j \mid \xi_t = \mathcal{D}_1) = \frac{p}{3}, \ \frac{1+p}{3}, \ \frac{2(1-p)}{3} \quad \text{for } j = 0,2,3 \text{ respectively.} \quad (39)$$

For any $t \in \mathbb{Z}^+$, conditional on $\xi_t \in \mathcal{D} \setminus \{\mathcal{D}_0, \mathcal{D}_1\}$, $|\xi_{t+1}| - |\xi_t|$ takes values only in $\{-1, 0, +1\}$ and for any $S \in \mathcal{D} \setminus \{\mathcal{D}_0, \mathcal{D}_1\}$

$$\mathbb{P}_{0,p}(|\xi_{t+1}| - |\xi_t| = 1 \mid \xi_t = S) = \frac{2(1-p)}{2N(S)+1}, \quad \text{and} \quad (40)$$

$$\mathbb{P}_{0,p}(|\xi_{t+1}| - |\xi_t| = -1 \mid \xi_t = S) = \frac{p(\mathbf{1}_{\{n_1(S)=1\}} + \mathbf{1}_{\{m_{N(S)}(S)=1\}})}{2N(S)+1} \leq \frac{2p}{2N(S)+1}. \quad (41)$$

Proof. The statements (38) and (39) are straightforward. Suppose that $\xi_t = S$ for some $S \in \mathcal{D} \setminus \{\mathcal{D}_0, \mathcal{D}_1\}$. Then $|S| \geq 2$ and exclusion moves cannot effect a change of magnitude more than 1. We have that $|\xi_{t+1}| = |S| + 1$ if and only if we select (with probability $2/(2N(S)+1)$) one of the two extreme 10 pairs, and then (with probability $1-p$) we flip the 10 to a 01. Similarly, $|\xi_t|$ can decrease by 1 if and only if there exists a configuration $\ldots 11101 \ldots$ at the left end or a configuration $\ldots 01000 \ldots$ at the right end, and then we select the 01 and flip to 10. The statement of the lemma follows. □

Lemma 17 *Suppose $\beta = 0$ and $p = 1/2$. Then*

$$\mathbb{E}_{0,1/2}[|\xi_{t+1}|^3 - |\xi_t|^3 \mid \mathcal{F}_t] \geq 4 \text{ a.s..} \quad (42)$$

Proof. First we have from (38) and (39) that

$$\mathbb{E}_{0,1/2}[|\xi_{t+1}|^3 - |\xi_t|^3 \mid \xi_t = \mathcal{D}_0] = 4, \quad \mathbb{E}_{0,1/2}[|\xi_{t+1}|^3 - |\xi_t|^3 \mid \xi_t = \mathcal{D}_1] = 5.$$

Thus it remains to consider the case where $\xi_t = S$ for $S \in \mathcal{D} \setminus \{\mathcal{D}_0, \mathcal{D}_1\}$. Here, from (40) and (41), writing $N = N(S)$, we have

$$\begin{aligned}\mathbb{E}_{0,1/2}[|\xi_{t+1}|^3 - |\xi_t|^3 \mid \xi_t = S] &\geq \frac{1}{2N+1}\left[\Big((|S|+1)^3 - |S|^3\Big) + \Big((|S|-1)^3 - |S|^3\Big)\right] \\ &= \frac{6|S|}{2N+1} \geq \frac{12N}{2N+1} \geq 4,\end{aligned}$$

since $|S| \geq 2N(S)$ and $N(S) \geq 1$ for all $S \neq \mathcal{D}_0$. □

Proof of Theorem 9. The upper bound in the theorem is implied by (3). For the lower bound, use Theorem 3.3 of [18] with, in the notation of that paper, $f(x) = x^3$ and $Y_n = |\xi_n|$. Then using (42) and the fact that $|\xi_t|$ has uniformly bounded jumps (see Lemma 16) we obtain the desired result. □

We now work towards the upper bound for $|\xi_t|$ for $p \in [0,1]$ given in Theorem 8. Define the function ρ^2 by $\rho^2(\mathcal{D}_0) := 0$ and for $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$

$$\rho^2(S) := \sum_{i=1}^{N} m_i^2 + \sum_{i=1}^{N} n_i^2 \ . \tag{43}$$

Lemma 18 *For any $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$,*

$$|S| \leq \frac{1}{2N(S)}|S|^2 \leq \rho^2(S) \leq |S|^2. \tag{44}$$

Proof. Suppose $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$. For the upper bound, we have $\rho^2(S) \leq \sum_{i=1}^{N}(m_i+n_i)^2 \leq |S|^2$. For the lower bound, $\frac{1}{N}\sum_{i=1}^{N} m_i^2 \geq \left(\frac{1}{N}\sum_{i=1}^{N} m_i\right)^2$ from Jensen's inequality, and similarly for the n_i. Hence

$$\rho^2(S) \geq \frac{1}{N}(R_N^2 + T_1^2) \geq \frac{1}{2N}(R_N + T_1)^2 = \frac{1}{2N}|S|^2 \geq |S|,$$

since $|S| \geq 2N$, completing the proof. □

Lemma 19 *Suppose $\beta = 0$ and $p \in [0,1]$. For $t \in \mathbb{Z}^+$ we have*

$$\mathbb{E}_{0,p}[\rho^2(\xi_{t+1}) - \rho^2(\xi_t) \mid \mathcal{F}_t] \leq 2 \text{ a.s.}; \tag{45}$$

moreover, for any $\varepsilon > 0$, $\mathbb{P}_{0,p}$-a.s., for all but finitely many $t \in \mathbb{Z}^+$

$$\max_{0 \leq s \leq t} \rho^2(\xi_s) \leq t(\log t)^{1+\varepsilon}. \tag{46}$$

Proof. We start by proving (45). First we note that for any $t \in \mathbb{Z}^+$

$$\mathbb{E}_{0,p}[\rho^2(\xi_{t+1}) - \rho^2(\xi_t) \mid \xi_t = \mathcal{D}_0] = 2(1-p) \leq 2.$$

We next need to verify (45) for any configuration $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$.

Let $\Delta_{1,i}(S)$ denote the change in $\rho^2(S)$ when a $01 \mapsto 10$ exclusion move is performed on the i-th 01 pair in S ($i = 1, \ldots, N$). Similarly let $\Delta_{2,i}(S)$ denote the change in $\rho^2(S)$ when a $10 \mapsto 01$ exclusion move is performed on the i-th 10 pair ($i = 0, \ldots, N$). Thus

$$\Delta_{1,i}(S) := \rho^2(e_i^{01\mapsto 10}(S)) - \rho^2(S); \quad \Delta_{2,i}(S) := \rho^2(e_i^{10\mapsto 01}(S)) - \rho^2(S),$$

in the notation of Section 4.2. Then

$$\mathbb{E}_{0,p}[\rho^2(\xi_{t+1}) - \rho^2(\xi_t) \mid \xi_t = S] = \frac{1}{2N+1}\left(p\sum_{i=1}^{N}\Delta_{1,i}(S) + q\sum_{i=0}^{N}\Delta_{2,i}(S)\right). \tag{47}$$

We compute the two sums on the right-hand side of (47) separately. Consider first all N possible exclusion moves $01 \mapsto 10$. Separating out the cases when $m_i = 1$ or $n_i = 1$,

$$\Delta_{1,i}(S) = -(2m_i - 2)\mathbf{1}_{\{m_i>1\}} - (2n_i - 2)\mathbf{1}_{\{n_i>1\}} + 2m_{i-1}\mathbf{1}_{\{n_i=1\}} + 2n_{i+1}\mathbf{1}_{\{m_i=1\}},$$

with the convention $n_{N+1} = m_0 = -1/2$, to make this formula correct for $i = 1$ and $i = N$. Since $(x-1)\mathbf{1}_{\{x>1\}} = x - 1$ for $x \in \mathbb{N}$ this last equation is

$$\Delta_{1,i}(S) = 2[2 - m_i - n_i + m_{i-1}\mathbf{1}_{\{n_i=1\}} + n_{i+1}\mathbf{1}_{\{m_i=1\}}].$$

Hence summing over $i \in \{1, \ldots, N\}$ gives

$$\frac{1}{2}\sum_{i=1}^{N} \Delta_{1,i}(S) = 2N - \sum_{i=1}^{N}(m_i + n_i) + \sum_{i=0}^{N-1} m_i \mathbf{1}_{\{n_{i+1}=1\}} + \sum_{i=2}^{N+1} n_i \mathbf{1}_{\{m_{i-1}=1\}} \leq 2N \ . \quad (48)$$

Similarly, a $10 \mapsto 01$ exclusion move on the i-th 10 pair ($i = 0, 1, \ldots, N$) contributes

$$\Delta_{2,i}(S) = 2\left[2 - m_i - n_{i+1} + m_{i+1}\mathbf{1}_{\{n_{i+1}=1\}} + n_i \mathbf{1}_{\{m_i=1\}}\right],$$

with the conventions $n_0 = m_{N+1} = 0$ and $n_{N+1} = m_0 = 1/2$ to make this formula correct for $i = 0$ and $i = N$. Summing, as before,

$$\frac{1}{2}\sum_{i=0}^{N} \Delta_{2,i}(S) = 2N + 1 - \sum_{i=1}^{N} m_i \mathbf{1}_{\{n_i>1\}} - \sum_{i=1}^{N} n_i \mathbf{1}_{\{m_i>1\}} \leq 2N + 1 \ . \quad (49)$$

Combining (48) and (49) with (47), we conclude that

$$\mathbb{E}_{0,p}[\rho^2(\xi_{t+1}) - \rho^2(\xi_t) \mid \xi_t = S] \leq \frac{2(2N+q)}{2N+1} \leq 2,$$

which is (45). Finally, (46) follows from (45) with Lemma 1(ii), taking $X_t = \rho^2(\xi_t)$. □

Suppose $p \in [0, 1]$. Then from (46), (37) and the middle inequality in (44) we obtain an upper bound for $\max_{0 \leq s \leq t} |\xi_s|$ of order $t^{3/4}$ (ignoring logarithmic factors). In order to prove the upper bound in Theorem 8(ii), we will give an argument that improves the 3/4 to 2/3. We start with a simple inequality.

Lemma 20 *Let $N \in \mathbb{N}$. Suppose that $n_1, n_2, \ldots, n_N \geq 0$. If, for some $A, B > 0$,*

$$\sum_{i=1}^{N} n_i^2 \leq A \quad \text{and} \quad \sum_{i=1}^{N} i n_i \leq B, \quad \text{then} \quad \sum_{i=1}^{N} n_i \leq (6AB)^{1/3}.$$

Proof. In the elementary inequality $\left(\sum_{i=1}^{N} n_i\right)^3 \leq 3\sum_{i=1}^{N} n_i^2 \sum_{i=1}^{N} n_i + 3\sum_{i=1}^{N} n_i \left(\sum_{j=1}^{i-1} n_j\right)^2$ apply Jensen's inequality to the final term to obtain

$$\left(\sum_{i=1}^{N} n_i\right)^3 \leq 3\sum_{i=1}^{N} n_i^2 \sum_{i=1}^{N} i n_i + 3\sum_{i=1}^{N} i n_i \sum_{j=1}^{i-1} n_j^2 \leq 6AB. \ \square$$

Proof of Theorem 8. Part (i) of the theorem is (37), and the lower bound in part (ii) of the theorem is (36). We now derive the upper bound in part (ii). Since each block of 1s has at least one element, observe that $\sum_{i=1}^{N} n_i(N-i) \leq f_1(S)$, and also $\sum_{i=1}^{N} n_i^2 \leq \rho^2(S)$. Thus Lemma 20 implies that for any $S \in \mathcal{D} \setminus \{\mathcal{D}_0\}$

$$\sum_{i=1}^{N} n_i \leq (6 f_1(S) \rho^2(S))^{1/3} \leq 2(f_1(S)\rho^2(S))^{1/3},$$

and the same argument applies for $\sum_{i=1}^{N} m_i$. Hence for any $S \in \mathcal{D}$

$$|S| \leq 4(f_1(S)\rho^2(S))^{1/3}. \quad (50)$$

Taking $S = \xi_t$, we have $f_1(\xi_t) \leq C_1 t$ for all t and some $C_1 \in (0, \infty)$ by (34). Also for any $\varepsilon > 0$, $\mathbb{P}_{0,p}$-a.s. $\rho^2(\xi_t) \leq C_2 t (\log t)^{1+\varepsilon}$ for all t by (46), for some $C_2 \in (0, \infty)$. Using these bounds in (50) completes the proof. □

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